Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Udai Pratad Singh, Udai Bir Singh

Geodesic torsion of a curve of a Fins1er hypersurface

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **55** (1973), n.3-4, p. 210–213.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_55_3-4_210_0>

Geometria differenziale. — *Geodesic torsion of a curve of a Finsler hypersurface.* Nota (*) di UDAI PRATAD SINGH e UDAI BIR SINGH, presentata dal Socio E. BOMPIANI.

RIASSUNTO. — Definizione e proprietà della torsione di una curva appartenente ad una ipersuperficie di uno spazio di Finsler.

1. NOTATIONS AND BASIC CONCEPTS

Consider a hypersurface F_{n-1}, given by equations $x^i = x^i(u^\alpha)$ $(i = 1, 2, \cdots n, \alpha = 1, 2, \cdots n-1)$ of an n-dimensional Finsler space F_n.

The angle τ between two vectors A^i and B^i of F_n is given by

$$\sin\tau = \frac{(\delta^{pq}_{ab}\, g_{ip}\, g_{jq}\, A^a\, B^b\, A^i\, B^j)^{1/2}}{(g_{ij}\, A^i\, A^j)^{1/2}\,(g_{ij}\, B^i\, B^j)^{1/2}} \tag{1.1}$$

where g_{ij} is the metric tensor of F_n and δ^{pq}_{ab} stands for $(\delta^p_a\,\delta^q_b - \delta^p_b\,\delta^q_a)$.

The unit normal vector $N^i(x, \dot{x})$ of the hypersurface satisfies the relations

$$N_i\, B^i_\alpha = 0 \quad \text{and} \quad g_{ij}\, N^i\, N^j = 1 \tag{1.2}$$

where $B^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$.

We mention below some important formulae which will be used in the later section of this paper.

The normal curvature vector $I^i_{\alpha\beta}$ satisfies the relation

$$I^i_{\beta\alpha} = \tilde{\Omega}_{\alpha\beta}\, N^i \tag{1.3}$$

where $\tilde{\Omega}_{\alpha\beta}$ are the components of the second fundamental tensor of the hypersurface.

The induced covariant derivative $N^i_{\|\gamma}$ has been given by [4]

$$N^i_{\|\gamma} = -\tilde{\Omega}_{\alpha\gamma}\, g^{\alpha\varepsilon}\, B^i_\varepsilon + (N^i\, M_l - 2\, M^i_l)\, I^l_{\sigma\gamma}\, \dot{u}^\sigma \tag{1.4}$$

where

$$M_{ij}(x, \dot{x}) = C_{ijh}(x, \dot{x})\, N^h(x, \dot{x}) \quad , \quad M_i(x, \dot{x}) = M_{ij}(x, \dot{x})\, N^j(x, \dot{x}) \tag{1.5}$$

$$M_{\alpha\beta}(u, \dot{u}) = M_{ij}(x, \dot{x})\, B^{ij}_{\alpha\beta} \quad , \quad M_\alpha(u, \dot{u}) = M_i(x, \dot{x})\, B^i_\alpha \tag{1.6}$$

$$M_{ij}(x, \dot{x})\, \dot{x}^j = 0 \quad , \quad M_i(x, \dot{x})\, \dot{x}^i = 0 \tag{1.7}$$

(*) Pervenuta all'Accademia il 15 settembre 1973.

and

$$M_{\alpha\beta}(u, \dot{u})\,\dot{u}^{\beta} = 0 \quad , \quad M_{\alpha}(u, \dot{u})\,\dot{u}^{\alpha} = 0 \tag{1.8}$$

$$\left(C_{ijh}(x, \dot{x}) = \frac{1}{2}\,\frac{\partial g_{ij}(x, \dot{x})}{\partial \dot{x}^{h}}\right)$$

The equation (1.4) may be used in defining

$$\frac{DN^{i}}{Ds} = N^{i}_{\|\gamma}\,\frac{du^{\gamma}}{ds} \tag{1.9}$$

The principal directions of F_{n-1} are given by

$$\tilde{\Omega}_{\alpha\beta}\,\dot{u}^{\alpha} = \underset{n}{\tilde{K}}\,g_{\alpha\beta}\,\dot{u}^{\alpha} \tag{1.10}$$

where $\underset{n}{\tilde{K}} = \tilde{\Omega}_{\alpha\beta}\,\dot{u}^{\alpha}\,\dot{u}^{\beta}$ is the normal curvature of F_{n-1}.

2. GEODESIC TORSION

Consider the vectors

$$\xi^{i}_{1|} = x^{|i} \quad , \quad \xi^{i}_{h|} = \frac{D\xi^{i}_{h-1|}}{Ds} \quad , \qquad (h = 2, \cdots, n). \tag{2.1}$$

These are used in defining ([3], pp. 153) mutually orthogonal vectors

$$n^{i}_{1|} = \xi^{i}_{1|} \quad , \quad n^{i}_{p|} = \frac{1}{\sqrt{D_{p-1}\,D_{p}}}\begin{vmatrix} (1,1) & \cdots & (1, p-1) & \xi^{i}_{1|} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p, p-1) & \xi^{0}_{p|} \end{vmatrix} \tag{2.2}$$

$$(p, q = 2, \cdots\cdots\cdots n)$$

where

$$D_0 = 1 \quad , \quad D_p = \begin{vmatrix} (1,1) & \cdots & (1,p) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ (p,1) & \cdots & (p,p) \end{vmatrix}$$

and

$$(p, q) = g_{ij}\,\xi^{i}_{p|}\,\xi^{j}_{q|}\,.$$

The Frenet's formulae for the curve are given by [3]

$$\frac{Dn^{i}_{p|}}{Ds} = -\frac{1}{\rho^{p-1}}\,n^{i}_{p-1|} + \frac{1}{\rho^{p}}\,n^{i}_{p+1|} \qquad (p = 2, \cdots, n-1) \tag{2.3}$$

where

$$\frac{1}{\rho^{p}} = \frac{\sqrt{D_{p-1}\,D_{p}}}{D_{p}}\left(\frac{1}{\rho^{0}} = \frac{1}{\rho^{n}} = 0\right) \qquad \text{for} \quad p = 1, \cdots, (n-1).$$

Assuming that the curve is a geodesic of F_{n-1} and using equation (2.1) one obtains

$$\xi^{i}_{2|} = k_{1|}\, N^{i} \tag{2.4}$$

where

$$k_{1|} = g_{ij} \frac{Dx^{|i}}{Ds} N^{j}.$$

If we further assume that the vectors ξ^{i}_{k1} (for $k = 1, \cdots, n$) are linearly independent then the equations (2.2) and (2.4) will yield

$$n^{i}_{2|} = N^{i}. \tag{2.5}$$

Hence the equation (2.3) for $p = 2$ reduces to

$$\frac{DN^{i}}{Ds} = -\frac{1}{\rho^{1}} \xi^{i}_{1|} + T_{g}\, n^{i}_{3|} \tag{2.6}$$

where T_{g} is employed to denote $1/\rho^{2}$. The scalar T_{g} is called the geodesic torsion of the curve.

3. Properties of Geodesic Torsion

Let w^{i} be unit vector along DN^{i}/Ds and ϑ be the angle between $n^{i}_{3|}$ and w^{i}. Using (2.6) we obtain

$$T_{g} = {}_{N}K_{G} \cos \vartheta \tag{3.1}$$

where ${}_{N}K_{G}$ is the magnitude of DN^{i}/Ds.

From (2.6) it is clear that w^{i} is linearly dependent on the orthogonal vectors $n^{i}_{3|}$ and $\xi^{i}_{1|}$. Hence if φ is the angle between w^{i} and ξ^{i}_{1} then

$$T_{g} = e\, {}_{N}K_{G} \sin \varphi \tag{3.2}$$

where

$$e = \pm 1.$$

The equations (1.1) (1.9) and (3.2) yield

$$T^{2}_{g} = \delta^{np}_{il}\, g_{nj}\, g_{ph}\, N^{i}_{||\gamma}\, N^{j}_{||\delta}\, \xi^{l}_{1|}\, \xi^{h}_{1|}\, \dot{u}^{\gamma}\, \dot{u}^{\delta}. \tag{3.3}$$

We shall now prove the following:

Theorem (3.1). *The geodesic torsion of a curve is given by*

$$T^{2}_{g} = \tilde{\Omega}_{\alpha\gamma}\, \tilde{\Omega}_{\beta\delta}\, g^{\alpha\beta}\, \dot{u}^{\gamma}\, \dot{u}^{\delta} + 4\, \tilde{\Omega}_{\beta\delta}\, M^{\beta}\, \dot{u}^{\delta}\, \underset{n}{\tilde{K}} + (4 M_{j} M^{j} - 1 - 3 M^{2})\, \underset{n}{\tilde{K}}^{2}.$$

Proof. Using equations (1.2), (1.3), (1.4), (1.5) and (1.6) the equation (3.3) reduces to

$$(3.4) \qquad T_g^2 = \tilde{\Omega}_{\alpha\gamma}\,\tilde{\Omega}_{\beta\delta}\,g^{\alpha\beta}\,\dot{u}^{\gamma}\,\dot{u}^{\delta} + 4\,\tilde{\Omega}_{\beta\delta}\,M^{\beta}\,\dot{u}^{\delta}\,\underset{n}{\tilde{K}} + (4\,M_j\,M^j - 1 - 3\,M^2)\,\underset{n}{K}^2$$

where

$$M^{\beta} = g^{\beta\varepsilon}\,M_{\varepsilon} \qquad \text{and} \quad M = M_j\,N^j,$$

The equation (3.4) expresses the geodesic torsion in terms of first and second fundamental tensors. If the curve is along a principal direction then with the help of equation (1.10) and (1.8) Theorem (3.1) reduces to

THEOREM (3.2). *The geodesic torsion of a line of curvature is*

$$(4\,M_j M^j - 3\,M^2)^{1/2}\,\underset{n}{\tilde{K}}.$$

Finally we have

THEOREM (3.3). *If the curve is non-asymptotic and is along the principal direction then the necessary and sufficient condition that its geodesic torsion vanishes is that*

$$4\,M_j M^j - 3\,M^2 = 0.$$

Proof. For a non-asymptotic curve $\underset{n}{K} \neq 0$.

From (3.2) it is clear that $T_g = 0$ iff

$$4\,M_j M^j - 3\,M^2 = 0.$$

This proves the theorem.

REFERENCES

[1] PAN T. K., *Indicatric torsion in a subspace of a Riemannian space*, « Proc. Amer. Math. Soc. », *8*, 294–298, April 1957.
[2] NIRMALA K., *Geodesic torsion of a congruence of curves*, « Tensor (N. S.) », *22* (3), 341–344 (1971).
[3] RUND H., *Differential Geometry of Finsler spaces*, Springer–Verlag, 1959.
[4] RUND K., *The Intrinsic and Induced curvature theories of subspaces of a Finsler space*, « Tensor (N. S.) », *16* (3), 294–312, September 1965.