Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Boundedness theorems for certain third order differential equations

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **55** (1973), n.3-4, p. 194–201.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_55_3-4_194_0>

Equazioni differenziali ordinarie. — *Boundedness theorems for certain third order differential equations.* Nota (*) di James O. C. Ezeilo e H. O. Tejumola, presentata dal Socio G. Sansone.

Riassunto. — Sono dimostrati due teoremi di limitatezza e di asintotica limitatezza per le soluzioni di due classi di equazioni differenziali non lineari del terzo ordine.

1. Introduction

We shall be concerned here with the uniform ultimate boundedness of solutions of the differential equation

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + h(x) = p(t), \tag{1.1}$$

where $a > 0$ is a constant and g, h, p depend only on the arguments shown. The function $h(x)$ is assumed differentiable and $g(x), h'(x), p(t)$ are continuous for all x and t.

The boundedness of solutions of (1.1) has been the subject of much study by several authors (see Chapter IV of [1] for a fairly comprehensive account of this). Lately Swick [2] generalizing a number of previously known results for the case in which $P(t) \equiv \int_0^t p(s)\,ds$ is bounded for all t, established uniform ultimate boundedness for solutions of (1.1) subject to the condition, only, that there are positive constants b, c with $ab > c$ such that $h'(x) \leq c$ for all x and such that, also,

$$G(x)/x \geq b \quad \text{and} \quad h(x)\,\mathrm{sgn}\,x \geq \eta, \qquad (|x| \geq R) \tag{1.2}$$

where $G(x) \equiv \int_0^x g(s)\,ds$ and η is a constant such that

$$\eta > \frac{1}{2}\,ca^{-1}. \tag{1.3}$$

Note that, as a result of the inequality $ab > c$, the condition on G in (1.2) is equivalent to saying that there is a constant $\gamma > 0$ such that

$$\{aG(x) - cx\}\,\mathrm{sgn}\,x \geq \gamma\,|x| \qquad (|x| \geq R). \tag{1.4}$$

Our main object in the present note is to show that the restriction (1.3) can be dispensed with altogether for the general equation (1.1), but we shall also

(*) Pervenuta all'Accademia il 10 ottobre 1973.

show, separately, for the special case

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + cx = p(t) \tag{1.5}$$

(corresponding to $h \equiv cx$ in (1.1)), with $c \equiv$ constant, that the condition (1.4) can be replaced by the much weaker condition:

$$\{a\mathrm{G}(x) - cx\} \operatorname{sgn} x \to +\infty \quad \text{as} \quad |x| \to \infty. \tag{1.6}$$

2. STATEMENT OF RESULTS

For the general equation (1.1) we shall establish the following.

THEOREM 1. *Suppose that there are positive constants* η, b, c *and* P_0, *with* $ab > c$, *such that*

(i) $|\mathrm{P}(t)| \leq \mathrm{P}_0$ *for all* t *considered,*

(ii) $h'(x) \leq c$ *for all* x,

(iii) h *and* G *satisfy* (1.2).

Then there exists a constant D_0 *whose magnitude depends only on* $\mathrm{R}, \mathrm{P}_0, a, b, c$ *and* g *such that every solution* $x(t)$ *of* (1.1) *satisfies*

$$|x(t)| \leq \mathrm{D}_0 \quad , \quad |\dot{x}(t)| \leq \mathrm{D}_0 \quad \textit{and} \quad |\ddot{x}(t)| \leq \mathrm{D}_0 \tag{2.1}$$

for all sufficiently large t.

For the special equation (1.5), in which c is assumed to be a positive constant, it will be shown simply that

THEOREM 2. *If* p *satisfies the condition* (i) *of Theorem 1 above, and if* G *satisfies* (1.6), *then every solution* $x(t)$ *of* (1.5) *ultimately satisfies* (2.1), *for some constant* D_0 *whose magnitude depends only on* P_0, a, c *and* g.

In what follows $\mathrm{D}, \mathrm{D}_1, \mathrm{D}_2, \cdots$ denote finite positive constants whose magnitudes depend only on $\mathrm{R}, \mathrm{P}_0, a, b, c$ and on g. The D's without suffixes are not necessarily the same each time they occur, but each of the numbered D's: $\mathrm{D}_1, \mathrm{D}_2, \mathrm{D}_3, \cdots$ retains a fixed identity throughout. Finally wherever it occurs, $\mathrm{D}(\varepsilon)$ denotes a constant whose magnitude depends on $\mathrm{R}, \mathrm{P}_0, a, b, c, g$ as well as on the quantity ε.

3. PROOF OF THEOREM 1

The procedure will be on the same lines as in [2] starting with the system-form of the equation (1.1):

$$\dot{x} = y \quad , \quad \dot{y} = z - ay - \mathrm{G}(x) + \mathrm{P}(t) \quad , \quad \dot{z} = -h(x), \tag{3.1}$$

except that we shall make use of a slightly modified Lyapunov function $\mathrm{V} = \mathrm{V}(x, y, z)$ given by

$$\mathrm{V} = \mathrm{V}_1 - \varepsilon \mathrm{V}_2, \tag{3.2}$$

where V_1 is as in [2], that is

$$V_1 = a \int_0^x h(s)\, ds + \beta \int_0^y G(s)\, ds + \frac{1}{2}(z^2 + \beta y^2) + yh(x) - \beta xz,$$

but V_2, different from Swick's V_2, is given by

$$V_2 = \begin{cases} (y + ax)\operatorname{sgn} z, & \text{if } |z| \geq |y + ax|, \\ z \operatorname{sgn}(y + ax), & \text{if } |z| \leq |y + ax|. \end{cases} \tag{3.3}$$

Here $\beta > 0$ is a constant fixed (as is possible, since $ab > c > 0$) such that

$$b > \beta > a^{-1} c, \tag{3.4}$$

and $\varepsilon > 0$ is an arbitrary constant.

It is clear from (3.3) that $|V_2| \leq |z|$ for all x, y and z. Also, Swick's estimates in [2; § 3] show quite clearly that

$$2 V_1 \geq (\beta x - z)^2 + \beta \{y + \beta^{-1} h(x)\}^2 + \beta(b - \beta) x^2 - D$$

and the expression on the right hand side here tends to $+\infty$ as $x^2 + y^2 + z^2 \to \infty$ since $(b - \beta) > 0$, by (3.4). Hence

$$V \to +\infty \qquad \text{as} \quad x^2 + y^2 + z^2 \to \infty \tag{3.5}$$

for each fixed ε. It remains now only to verify the other Lyapunov property involving the function

$$\dot{V}^*(t) \equiv \limsup_{h \to +0} \frac{V(x(t+h), y(t+h), z(t+h)) - V(x(t), y(t), z(t))}{h}$$

corresponding to any solution $(x(t), y(t), z(t))$ of (3.1). What we shall in fact show here is that, if ε is fixed sufficiently small (more precisely: $\varepsilon \leq D$, with D sufficiently small) then

$$\dot{V}^*(t) \leq -1 \qquad \text{whenever} \quad x^2(t) + y^2(t) + z^2(t) \geq D_1^2, \tag{3.6}$$

for some D_1.

Let then $(x, y, z) = (x(t), y(t), z(t))$ be any solution of (3.1). It is an elementary matter to verify from the definitions of V, V_1 and V_2 that

$$\dot{V}^* = -\{G(x) - \beta x - P(t)\} h(x) + W_1 - \varepsilon W_2$$

where

$$W_1 = -(a\beta - h'(x)) y^2 + \beta P(t) y$$

$$W_2 = \begin{cases} |z| + [P(t) - G(x)] \operatorname{sgn} z, & \text{if } |z| \geq |y + ax|, \\ -h(x) \operatorname{sgn}(y + ax), & \text{if } |z| \leq |y + ax|. \end{cases}$$

Since $h'(x) \leq c$ and $a\beta - h'(x) \geq a\beta - c > 0$, by (3.4), it is evident that

$$\begin{aligned} W_1 &\leq -2 D_2 y^2 + \beta P_0 |y| \\ &\leq -D_2 y^2 + D \end{aligned}$$

where $D_2 = \frac{1}{2}(a\beta - c)$. The foregoing estimates show clearly that

$$\text{(3.7)} \qquad \dot{V}^* \leq -D_2 y^2 - [G(x) - \beta x] h(x) + \varepsilon |G(x)| + (P_0 + \varepsilon) |h(x)| + D(\varepsilon)$$

always, but that

$$\text{(3.8)} \qquad \dot{V}^* \leq -\varepsilon |z| - D_2 y^2 - [G(x) - \beta x] h(x) + \varepsilon |G(x)| + P_0 |h(x)| + D(\varepsilon), \quad \text{if} \quad |z| \geq |y + ax|.$$

Now, if $|x| \geq R$, then

$$\text{(3.9)} \qquad [G(x)/x] - \beta \geq b - \beta > 0,$$

by (1.2) and (3.4), so that

$$\begin{aligned} & -[G(x) - \beta x] h(x) + \varepsilon |G(x)| + (P_0 + \varepsilon) |h(x)| \\ = & -|G(x) - \beta x| \cdot |h(x)| + \varepsilon |G(x)| + (P_0 + \varepsilon) |h(x)| \\ \leq & -|G(x) - \beta x| \cdot |h(x)| + \varepsilon |G(x) - \beta x| + \varepsilon\beta |x| + (P_0 + \varepsilon) |h(x)| \\ \equiv & -\frac{1}{4} |G(x) - \beta x| \cdot |h(x)| + U_1 + U_2 + U_3 \end{aligned}$$

where

$$U_1 = -\frac{1}{4} \{|G(x) - \beta x| - 4(P_0 + \varepsilon)\} |h(x)|$$

$$U_2 = -\frac{1}{4} \{|h(x)| - 4\varepsilon\} |G(x) - \beta x|$$

$$U_3 = -\frac{1}{4} \{|G(x) - \beta x| \cdot |h(x)| - 4\varepsilon\beta |x|\}.$$

But, by (1.2) and (3.9),

$$U_1 \leq -\frac{1}{4} \eta [(b - \beta) |x| - 4(P_0 + \varepsilon)] \leq 0$$

if $|x| \geq D \geq R$, for arbitrary ε. Also

$$U_2 \leq -\frac{1}{4} (\eta - 4\varepsilon)(b - \beta) |x| \leq 0$$

if $|x| \geq R$ and $\varepsilon \leq \frac{1}{4}\eta$. Finally

$$U_3 \leq -\frac{1}{4} \{\eta(b - \beta) - 4\varepsilon\beta\} |x| \leq 0$$

if $|x| \geq R$ and $\varepsilon \leq \frac{1}{4} \eta\beta^{-1}(b - \beta)$. Hence if

$$\text{(3.10)} \qquad \varepsilon \leq \frac{1}{4} \eta \min\{1, \beta^{-1}(b - \beta)\},$$

as we shall henceforth assume, then there exists $D_3 \geq R$ such that

$$(3.11) \qquad -\{G(x)-\beta x\}\, h(x)+\varepsilon\,|G(x)|+(P_0+\varepsilon)\,|h(x)| \leq -$$
$$-\frac{1}{4}\,|G(x)-\beta x|\cdot|h(x)| \leq -\frac{1}{4}(b-\beta)\,\eta\,|x|,$$

if $|x| \geq D_3$, which when combined with (3.7) shows that

$$(3.12) \qquad \dot{V}^* \leq -1, \quad \text{if} \quad x^2(t)+y^2(t) \geq D_4^2$$

for some $D_4 \geq D_3$.

For the case when $x^2(t)+y^2(t) < D_4^2$, the estimate (3.8) is applicable provided that $|z(t)| \geq (a+1)\,D_4$; that is

$$\dot{V}^* \leq -\varepsilon\,|z|+D_5, \quad \text{if} \quad x^2(t)+y^2(t) < D_4^2 \quad \text{and} \quad |z(t)| \geq (a+1)\,D_4.$$

Thus

$$(3.13) \qquad \dot{V}^* \leq -1 \quad \text{if} \quad x^2(t)+y^2(t) < D_4^2$$

provided that $|z(t)| \geq D_6 \geq \max\{(a+1)\,D_4,\,(D_5+1)\,\varepsilon^{-1}\}$.

The two estimates (3.12) and (3.13) show that

$$\dot{V}^* \leq -1, \quad \text{if} \quad x^2(t)+y^2(t)+z^2(t) \geq D_4^2+D_6^2,$$

with ε subject to (3.10), which proves (3.6) and thus concludes our verification of Theorem 1.

4. Proof of theorem 2

This time it is convenient to take the differential equation (1.5) in the system-form:

$$(4.1) \qquad \dot{x}=z-ax \quad , \quad \dot{y}=-cx \quad , \quad \dot{z}=y-G(x)+P(t)$$

and to work with the Lyapunov function $V=V(x,y,z)$ given by

$$V=V_1-(P_0+2)\,V_2-(P_0+1)\,V_3$$

where

$$(4.2) \qquad \begin{cases} V_1=\displaystyle\int_0^x G(s)\,ds-xy+\frac{1}{2}(z^2+ac^{-1}y^2) \\ V_2=\begin{cases} z\,\operatorname{sgn} y, & \text{if} \quad |y| \geq |z|, \\ y\,\operatorname{sgn} z, & \text{if} \quad |z| \geq |y|, \end{cases} \\ V_3=\begin{cases} x\,\operatorname{sgn} z, & \text{if} \quad |z| \geq |x|, \\ z\,\operatorname{sgn} x, & \text{if} \quad |x| \geq |z|. \end{cases} \end{cases}$$

Note that V_1 can be reset in the form

$$V_1 = \int_0^x \{G(s) - a^{-1} cs\} \, ds + \frac{1}{2} \{(a^{1/2} c^{-1/2} y - c^{1/2} a^{-1/2} x)^2 + z^2\}$$

in which the integral

$$\int_0^x \{G(s) - a^{-1} cs\} \, ds \to +\infty \qquad \text{as} \quad |x| \to \infty.$$

because of (1.6). Thus, since $|V_2| \leq |z|$ and $|V_3| \leq |z|$ for all x, y and z, we have as before that

$$V \to +\infty \qquad \text{as} \quad x^2 + y^2 + z^2 \to \infty.$$

Next let $(x, y, z) \equiv (x(t), y(t), z(t))$ be any solution of (4.1). Then a simple calculation from (4.1) and (4.2), followed by the use of the condition that $|P(t)| \leq P_0$, will show that

$$\dot{V}^*(t) \leq -ax\{G(x) - a^{-1} cx\} + P_0 |z| + M_2 + M_3,$$

where

$$M_2 \leq \begin{cases} -(P_0 + 2)|y| + D(|G(x)| + 1), & \text{if} \quad |y| \geq |z|; \\ D|x|, & \text{if} \quad |z| \geq |y|, \end{cases}$$

$$M_3 \leq \begin{cases} -(P_0 + 1)|z| + D|x|, & \text{if} \quad |z| \geq |x|, \\ (P_0 + 1)(|y| + |G(x)| + P_0), & \text{if} \quad |x| \geq |z|. \end{cases}$$

Hence

$$\text{(4.3)} \quad \dot{V}^* \leq \begin{cases} -ax[G(x) - a^{-1}cx] - |z| - (P_0 + 2)|y| + D(|G(x)| + |x| + 1), \\ \qquad\qquad \text{if} \quad |y| \geq |z| \geq |x|, \\ -ax[G(x) - a^{-1}cx] - |y| + D(|G(x)| + |x| + 1), \\ \qquad\qquad \text{if} \quad |y| \geq |z| \quad \text{and} \quad |x| \geq |z|, \\ -ax[G(x) - a^{-1}cx] - |z| + D|x|, \quad \text{if} \quad |z| \geq |y| \quad \text{and} \quad |z| \geq |x|, \\ -ax[G(x) - a^{-1}cx] + D(|G(x)| + |x| + 1), \quad \text{if} \quad |x| \geq |z| \geq |y|. \end{cases}$$

Thus

$$\text{(4.4)} \qquad \dot{V}^* \leq -ax[G(x) - a^{-1}cx] + D_7|G(x)| + D_8|x| + D,$$

always, for some constants D_7 and D_8. But, by (1.6), there exists D_9 such that $[G(x) - a^{-1}cx\} \operatorname{sgn} x > 0$, so that

$$x[G(x) - a^{-1}cx] = |x| \cdot |G(x) - a^{-1}cx|,$$

for $|x| \geq D_9$. Hence, if $|x| \geq D_9$, we have from (4.4) that

$$\begin{aligned}\dot{V}^* &\leq -a|x|\cdot|G(x) - a^{-1}cx| + D_7|G(x)| + D_8|x| + D \\ &\leq -a|x|\cdot|G(x) - a^{-1}cx| + D_7|G(x) - a^{-1}cx| + D_{10}|x| + D \\ &\equiv -\frac{1}{2}a|x||G(x) - a^{-1}cx| + M_4 + M_5 + D\end{aligned}$$

where $D_{10} = D_8 + a^{-1}cD_7$ and

$$M_4 = -\frac{1}{4}(a|x| - 4D_7)|G(x) - a^{-1}cx|,$$

$$M_5 = -\frac{1}{4}(|aG(x) - cx| - 4D_{10})|x|.$$

Clearly $M_4 \leq 0$ and, by (1.6), $M_5 \leq 0$, if $|x|$ is sufficiently large. Hence there exists $D_{11} \geq D_{10}$ such that, if $|x| \geq D_{11}$, then

$$\begin{aligned}\dot{V}^* &\leq -\frac{1}{2}a|x||G(x) - a^{-1}cx| + D \\ &\leq -1\end{aligned} \tag{4.5}$$

provided, further, that $|x| \geq D_{12}$ with $D_{12} (\geq D_{11})$ sufficiently large.

It remains now to estimate $\dot{V}^*$ for $|x| \leq D_{12}$, and we shall consider only the case $y^2 + z^2$ large. If, for instance, $y^2 + z^2 \geq 2D_{12}^2$ then two distinct possibilities arise, namely: (I) $|z| \geq D_{12}$ or (II) $|z| < D_{12}$ and $|y| > D_{12}$. Always recalling that $|x| \leq D_{12}$ is assumed here, case (I) then implies that $|z| \geq |x|$ so that, by (4.3),

$$\dot{V}^* \leq \begin{cases} -|z| - (P_0 + 2)|y| + D, & \text{if } |y| \geq |z| \\ -|z| + D, & \text{if } |z| \geq |y| \end{cases}$$

which shows clearly that

$$\dot{V}^* \leq -1 \quad \text{if} \quad \min(|y|, |z|) \geq D_{13},$$

with $D_{13} (\geq D_{12})$ sufficiently large. Case (II) implies that $|z| < |y|$, so that again by (4.3),

$$\dot{V}^* \leq -|y| + D \leq -1$$

for sufficiently large $|y|$. The last two estimates for $\dot{V}^*$ show clearly that there is a constant D_{14} such that

$$\dot{V}^* \leq -1 \quad \text{if} \quad |x| \leq D_{12} \quad \text{and} \quad y^2 + z^2 \geq D_{14}^2$$

which, when combined with (4.5), shows that

$$\dot{V}^* \leq -1 \quad \text{if} \quad x^2(t) + y^2(t) + z^2(t) \geq D_{12}^2 + D_{14}^2$$

and the theorem now follows, as before.

5. A FURTHER GENERALIZATION OF THEOREM 2

It is possible to extend the conclusion of Theorem 2 to the slightly perturbed equation:

$$\dddot{x} + a\ddot{x} + g(x)\dot{x} + cx = p(t) + q(t, x, \dot{x}, \ddot{x}) \tag{5.1}$$

with a, g, c, p exactly as in Theorem 2, where q is a continuous function dependent on all the arguments shown and $|q(t, x, y, z)| \leq Q_0$ (constant) for all t, x, y, z. For the proof it will be necessary to take (5.1) in the system form

$$\dot{x} = z - ax \quad , \quad \dot{y} = -cx + q^* \quad , \quad \dot{z} = y - G(x) + P(t) ,$$

where

$$q^* \equiv q(t, x, z - ax, y + a^2 x - az - G(x) + P(t)),$$

and to use the function V given by

$$V = V_1 - \alpha V_2 - \beta V_3$$

where V_1, V_2, V_3 are exactly as in § 4, and

$$\alpha = P_0 + 2(1 + ac^{-1} Q_0) \quad , \quad \beta = P_0 + 1 + ac^{-1} Q_0 .$$

The other relevant details are as in § 4, and will therefore be omitted.

REFERENCES

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