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Ludvik Janos

An application of combinatorial techniques to a topological problem

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Topologia. — *An application of combinatorial techniques to a topological problem.* Nota (*) di LUDVIK JANOS, presentata dal Socio G. SANSONE.

RIASSUNTO. — Sia X un insieme avente al più la potenza del continuo e sia $f: X \to X$ una trasformazione tale che ogni iterata f^n ($n = 1, 2 \cdots$) ha un sol punto fisso. Allora per ogni $c \in (0, 1)$ esiste una metrica ρ su X tale che lo spazio metrico (X, ρ) è separabile ed f è una contrazione di costante c.

I. INTRODUCTION

In recent two decades different mathematicians asked the following question: Given an abstract set X and a mapping $f: X \to X$, does there exist a non-trivial topology on X which would render f continuous and would satisfy at the same time some prescribed conditions (compactness, separability, metrizability, Hausdorff property, etc.)? J. de Groot and H. de Vries [1] proved that if X has at most continuously many elements then for every $f: X \to X$ there exists a non-discrete separable metric topology on X rendering f continuous. C. Bessaga [2] obtained the following result (a converse to the Banach fixed point theorem).

THEOREM 1 (C. Bessaga). *Let* X *be a set and* $f: X \to X$ *such that all the iterates* f^n *have a unique fixed point. Assuming the weak (countable) form of the axiom of choice, then for any* $c \in (0, 1)$ *there exists a complete metric on* X *rendering* f *a c–contraction.*

The purpose of this note is to show that in case X has at most continuously many elements then the separability of the metric in the above theorem can be claimed. In the construction of this metric we will use the following combinatorial theorem of F. P. Ramsey [3].

THEOREM 2 (F. P. Ramsey). *If the set of all unordered pairs* $\{n, m\}$ *of natural numbers* N *is decomposed in finite number of sets, say* $R_1, R_2, \cdots, \cdots, R_k$, i.e.

$$\{A \mid |A| = 2 \text{ and } A \subset N\} = R_1 \cup R_2 \cdots \cup R_k$$

then there exists an infinite subset $M \subset N$ *and an index* $i \in \{1, 2, \cdots, k\}$ *such that all pairs* $\{n, m\} \subset M$ *belong to* R_i.

Finally we will need the following result of Ph. Meyers [4].

THEOREM 3 (Ph. Meyers). *If* X *is a metrizable topological space and* $f: X \to X$ *a continuous mapping satisfying*:

(*i*) *f has a unique fixed point a, i.e.* $f(a) = a$,

(*) Pervenuta all'Accademia il 20 agosto 1973.

(*ii*) *for every* $x \in X$ *the sequence of iterates* $x, f(x), f^2(x), \cdots$ *converges to* a;

(*iii*) *there exists a neighbourhood* U_a *of* a *such that for any neighbourhood* V_a *of* a *there exists* n_0 *such that* $n \geq n_0$ *implies* $f^n(U_a) \subset V_a$.

Then for every $c \in (0, 1)$ *there exists a metric on* X *which is compatible with the topology of* X *and with respect to which* f *is a* c*–contraction.*

II. PROOF OF THE THEOREM

Let X be an abstract set with at most continuously many elements and let $f: X \to X$ satisfy the conditions of the Theorem 1. Choosing $c = 1/2$ we denote by ρ the corresponding metric on X existing by this theorem. If a is the fixed point of f we define the sets A_n (n integer) by:

$$A_n = \{x \mid x \in X \text{ and } 2^{n-1} < \rho(a, x) \leq 2^n\}.$$

Thus we obtain a disjoint partition of X in the form $X = \{a\} \cup \bigcup_{-\infty}^{+\infty} A_n$ satisfying the condition that the image $f(A_n)$ of A_n under f is contained in $\{a\} \cup \bigcup_{-\infty}^{n-1} A_k$. Once achieved this result we disregard the metric ρ (since it is not separable in general) and proceed in the following way:

We consider the subset $\{0\} \cup \bigcup_{-\infty}^{+\infty} C_n$ of the Euclidean plane where o is the origin and C_n is the circle with centre in o and of radius 2^n. Since each set A_n has at most continuously many elements one can identify A_n with a certain subset $B_n \subset C_n$ of C_n. Doing this for every n and identifying a with the origin o, our set X can be thought of as the set $\{0\} \cup \bigcup_{-\infty}^{+\infty} B_n$. Denoting by d_2 the Euclidean metric we thus obtain a separable metric space (X, d_2) and it follows from the definition that each subset $\{0\} \cup \bigcup_{-\infty}^{n} B_k$ is totally bounded and invariant under f.

We now define a new metric d_2^* on X with respect to which f will be continuous as follows:

$$d_2^*(x, y) = \sup_{n \geq 0} d_2(f^n(x), f^n(y)),$$

for $x, y \in X$ and where $f^0(x)$ stands for x. It is clear that d_2^* is a metric and that f is continuous with respect to d_2^*, since from the definition it follows immediately that f is non-expanding:

$$d_2^*(f(x), f(y)) \leq d_2^*(x, y).$$

Since the circles C_n shrink to o it follows that for each pair $x, y \in X$ there is a number $n = n(x, y)$ such that $d_2^*(x, y) = d_2(f^n(x), f^n(y))$. In order to show that the sets $\{0\} \cup \bigcup_{-\infty}^{n} B_k$ are totally bounded also with respect to the metric d_2^* we need the following.

LEMMA. *Let* (Y, d) *be a totally bounded metric space and let* $f: Y \to Y$ *(not necessarily continuous) be such that the diameters* δ_n *of the iterated images* $f^n(Y)$ *converge to zero as* $n \to \infty$. *Then the metric* d^* *on* Y *defined by*

$$d^*(x, y) = \sup_{n \geq 0} d(f^n(x), f^n(y))$$

is also totally bounded.

Proof. First we observe that due to $\delta_n \to 0$ there is an integer $n = n(x, y)$ for each pair of points $x, y \in Y$ such that $d^*(x, y) = d(f^n(x), f^n(y))$. Now if d^* were not totally bounded there would be a number $\varepsilon > 0$ and a sequence $\{x_k\} \subset Y$ such that

$$d^*(x_k, x_l) \geq \varepsilon \quad \text{for all } k \neq l.$$

But this would mean that there is a function $n(k, l)$ on the set of all unordered pairs $\{k, l\}$ of natural numbers such that $d(f^{n(k,l)}(x_k), f^{n(k,l)}(x_l)) \geq \varepsilon$ for all pairs $\{k, l\} \subset N$. Again due to the shrinkage $\delta_n \to 0$ it is obvious that the function $n(k, l)$ must be bounded and so its range consists of finite numbers of values, say $n_1, n_2, \ldots, n_r$. But Theorem 2 would then imply that for some $i \in \{1, 2, \cdots, r\}$ the inequality $d(f^{n_i}(x_k), f^{n_i}(x_l)) \geq \varepsilon$ would hold for some infinite subset of indices which would contradict the assumption that d is totally bounded. This proves that d^* must be totally bounded as well.

Observing that the restriction of $f: X \to X$ to the invariant subset $X_n = \{0\} \cup \bigcup_{-\infty}^{n} B_k$ satisfies the hypothesis of our lemma we arrive at the following conclusion:

As a countable union of totally bounded sets, (X, d_2^*) is a separable metric space and $f: X \to X$ a continuous mapping. Since $d_2^* \geq d_2$ it follows that the topology generated by d_2^* is in general finer than the Euclidean generated by d_2. Since each set X_n is d_2-open, it is also d_2^*-open and observing that for each $x \in X$ we have $d_2^*(0, x) = d_2(0, x)$ it follows that each open neighbourhood of 0 with respect to d_2^* contains some set X_n. Since $f(X_n) \subset X_{n-1}$ this implies that the conditions of Theorem 3 are satisfied for the topology generated by d_2^* and our theorem follows from Theorem 3.

Remark. It is so far not known if the space (X, d_2^*) can be assumed *topologically* complete. In this case the result of Ph. Meyers [4] would furnish at the same time separable and complete metric. So it appears that the gain of separability was paid by the loss of completeness.

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