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Extreme invariant operators

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Analisi funzionale. — *Extreme invariant operators.* Nota [(*)] di SIMEON REICH, presentata dal Socio G. SANSONE.

RIASSUNTO. — Dato un semigruppo ammissibile a sinistra che opera su uno spazio di Banach E, si considèra il problema di caratterizzare gli elementi estremi della sfera unitaria degli operatori lineari invarianti che mostrano E in un altro spazio di Banach.
Sono date soluzioni per gli spazi L e per certi spazi di Lindenstrauss.

1. INTRODUCTION

In this Note, we are given a left amenable (topological) semigroup which acts on a Banach space E (see the definitions below). Our purpose is to characterize the extreme elements in the unit ball of invariant linear operators which map E into another Banach space. Thus our note can be considered a sequel to [2], although our method of attack is different. At the same time, our results are applications (and extensions) of recent theorems obtained by Fakhoury [9, 10] and Sharir [24, 25]. We shall confine our attention to L-spaces and certain Lindenstrauss spaces. [4] and [5] contain information on Banach spaces and amenable semigroups. All the Banach spaces considered in this paper are assumed to be over the reals.

2. PRELIMINARIES

A semigroup is a set with an associative binary operation $(s, t) \to st$. A topological semigroup is a semigroup with a Hausdorff topology in which the product st is separately continuous. A semigroup can always be made into a topological semigroup by endowing it with the discrete topology.

Let S be a topological semigroup, and let C (S) denote the Banach space of all bounded continuous real-valued functions on S with the supremum norm. For t in S and f in C (S), define $l_t f$, the left translate of f by t, by $l_t f(s) = f(ts)$, $s \in S$. The right translate $r_t f$ is defined by $r_t f(s) = f(st)$. A function f in C (S) is said to be left uniformly continuous if the map $s \to l_s f$ is continuous on S into C (S). We shall denote by LUC (S) the space of all left uniformly continuous functions on S [20, 18]. LUC (S) is a translation invariant closed subspace of C (S). S will be called left amenable if LUC (S) admits a left invariant mean. That is, if there exists a continuous linear functional m on LUC (S), of unit norm, which is positive and satisfies $m(l_t f) = m(f)$ for all t in S and f in LUC (S) [20, p. 67].

(*) Pervenuta all'Accademia il 27 luglio 1973.

The unit ball of a Banach space E will be denoted by $B(E)$. If F is another Banach space, then $L(E, F)$ will denote the Banach space of all linear continuous operators from E into F. We shall write $B(E, F)$ instead of $B(L(E, F))$. E^* will stand for the conjugate space of E. The set of extreme points of a subset Q of E will be denoted by ext Q.

Let S be a topological semigroup and E a Banach space. We shall say that S acts on E from the right if there is a separately continuous map $E \times S \to E$, denoted by $(x, s) \to xs$, such that the operator $x \to xs$ belongs to $B(E, E)$ for all $s \in S$ and $x(st) = (xs)t$ for all x in E and s, t in S.

Let F be a Banach space. An operator T in $L(E, F)$ is said to be invariant if $T(xs) = Tx$ for all x in E and s in S. Set $B(E, F)_S = \{T \in B(E, F) : T$ is invariant$\}$. Let T belong to $B(E, F)_S$ and let T^* be its adjoint. If $T^* y$ is an extreme point of $B(E^*)_S$ for each y in ext $B(F^*)$, then we shall say that T is invariantly nice. If $T^* y$ belongs to ext $B(E^*)_S$ for each y in a dense (with respect to the weak star topology) subset of ext $B(F^*)$, then T will be called almost invariantly nice. If S is the identity, "invariantly" will be omitted [19, p. 185]. The Krein-Milman theorem implies that if an operator T in $B(E, F)_S$ is almost invariantly nice, then it is an extreme point of $B(E, F)_S$. Although $E^*_S = \{y \in E^* : y$ is invariant$\}$ may, of course, be trivial, we shall not indicate this possibility explicitly in the sequel.

An L-space is a Banach lattice whose norm is additive on the positive cone. A Banach space is called a Lindenstrauss space [7, p. 435] if E^* is (isometric to) an L-space. The definitions of the Lindenstrauss spaces we shall consider can be found in [15, p. 180].

3. Amenable semigroups

We begin by stating a few lemmas.

LEMMA 1. [20, p. 68]. *If a left amenable topological semigroup* S *acts on a Banach space* E *from the right, then* E^*_S *is the range of a contractive projection* $Q : E^* \to E^*$.

Proof. If $x \in E$ and $y \in E^*$, then the function $b(x, y)$ defined on S by $b(s) = \langle xs, y\rangle$ is in LUC (S). Therefore, given a functional y in E^*, we can define a point Qy in E^* by $\langle x, Qy\rangle = m(b(x, y))$ where m is a left invariant mean on LUC (S). Clearly $\|Qy\| \leq \|y\|$. If y is invariant, then $b(s) = \langle x, y\rangle$ for all s, and therefore $Qy = y$. Finally, each Qy is invariant, because $\langle xt, Qy\rangle = m(b(xt, y)) = m(l_t b(x, y)) = m(b(x, y)) = \langle x, Qy\rangle$ for all t in S.

LEMMA 2. *Let* E *be a Banach space. If* $P : E \to E$ *is a contractive projection, then* $(PE)^*$ *is isometric to* $P^* E^*$, *and the weak star topology induced on* $P^* E^*$ *by* PE *agrees with the weak star topology inherited from* E^*.

Proof. Define $T : (PE)^* \to P^* E^*$ by

$$\langle x , Tz \rangle = \langle Px , z \rangle \quad \text{where} \quad x \in E \quad \text{and} \quad z \in (PE)^*.$$

T is an isometry onto. Now suppose that $\langle x , y_a \rangle \to 0$ for all $x \in PE$, where $\{y_a\}$ is a net in $P^* E^*$, and let $w \in E$. Then $\langle w , y_a \rangle = \langle w , P^* y_a \rangle = \langle Pw , y_a \rangle \to 0$.

A finite mean on LUC (S) is a mean which belongs to the convex hull of the set of point functionals on LUC (S).

LEMMA 3. *Let a topological semigroup* S *act on a Banach space* E *from the right. Suppose that* LUC (S) *admits a left invariant finite mean* m. *If we construct the projection* Q *of Lemma* 1 *with the aid of this* m, *then* Q *is the adjoint of a projection* $P : E \to E$.

Proof. In this case, it is easy to see that Q is weak star continuous. Therefore there is a map $P : E \to E$ such that $Q = P^*$. P must be a projection.

We shall say that the right action of a topological semigroup S on a Banach space E is weakly almost periodic [6, p. 72] if $\{xs : s \in S\}$ is weakly relatively compact for each $x \in E$.

LEMMA 4. *If the right action of a left amenable topological semigroup* S *on a Banach space* E *is weakly almost periodic, then the projection* Q *constructed in Lemma* 1 *is the adjoint of a projection* $P : E \to E$.

Proof. Let $Q^* : E^{**} \to E^{**}$ be the adjoint of Q and $\hat{E}$ the canonical image of E in E^{**}. Let x belong to E. In order to show that $Q^* \hat{x}$ is in $\hat{E}$, it is sufficient to prove that $Q^* \hat{x}$ is continuous when the Mackey topology $\tau (E^*, E)$ is imposed on E^*. Indeed let the net $\{y_a\} \subset E^*$ converge to zero in this topology. Then $\{y_a\}$ converges uniformly on weakly relatively compact subsets of E. Therefore $\langle y_a , Q^* \hat{x} \rangle = \langle Q y_a , \hat{x} \rangle = \langle x , Q y_a \rangle = m (b (x , y_a)) \to 0$ because the action of S is weakly almost periodic. Thus we may define a projection $P : E \to E$ by $\widehat{Px} = Q^* \hat{x}$. It is clear that $P^* = Q$.

THEOREM 1. *Let* E *and* F *be two* L-*spaces and* S *a left amenable topological semigroup which acts on* E *from the right. Then* $T \in B(E , F)_S$ *is extreme there if and only if it is invariantly nice.*

Proof. E^* can be identified with C (K) for some extremally disconnected K (Kakutani). The "into" extension property of such spaces [4, p. 94] implies that the range of the projection Q of Lemma 1 can be identified with C (X) for some extremally disconnected X. Simultaneously, this range is weak star closed in E^*. Therefore it is the conjugate space of an L-space G (which induces the given weak star topology) [12, p. 554]. Thus $B(E , F)_S$ may be identified with B (G , F). This completes the proof because Sharir [25] has shown that an extreme operator in B (G , F) must be nice.

A similar result can be proved for positive operators (cfr. [3, p. 204] and Theorem 3).

THEOREM 2. *Let* E *and* F *be two* C_σ *spaces and assume that* E *is separable. Let a left amenable topological semigroup* S *act on* E *from the right. Suppose that either*

(1) LUC (S) *admits a left invariant finite mean,*

or

(2) *The action of* S *on* E *is weakly almost periodic.*

Then $T \in B(E, F)_S$ *is extreme there if and only if it is almost invariantly nice.*

Proof. The previous lemmas imply that $B(E, F)_S$ can be identified with B (PE , F) where $P : E \to E$ is a contractive projection. PE is a separable C_σ space [16, p. 341]. An appeal to [9, Theorem 11] (or to [10, Theorem 14]) concludes the proof.

The separability assumption is probably redundant. It can be replaced by several other assumptions (cfr. [24]). On the other hand, "almost invariantly nice" cannot be replaced by "invariantly nice" even when both E and F are C (K) spaces. To see this, let G be a C_σ space which is not a C_Σ space, $K = \text{ext } B(G^*) \cup \{0\}$ with the weak star topology, and $E = F = C(K)$. Let the semigroup S consist of the identity i and t, where $t^2 = i$. If $x \in C(K)$, define $xi = x$, $xt(k) = -x(-k)$, and T : E. E by $Tx = \frac{1}{2}(x + xt)$. T is extreme in $B(E, F)_S$ (it is almost invariantly nice), but it is not invariantly nice. When conditions (1) and (2) are omitted, ${E^*}_S$ is still a conjugate L-space whenever E is a Lindenstrauss space. (This follows from Lemma 1). But its predual may turn out to be, for instance, an A (S) space, even when E is a C (K) space (see the example, due to Choquet, which is described in [22, p. 83]). This phenomenon disrupts our approach because an extreme operator in B (A (S) , F) need not be almost nice even if F is a C (K) space [13, p. 41].

The set of positive elements in a subset Q of an ordered Banach space will be denoted by Q^+. The action of a topological semigroup S on E will be called positive if the operator $x \to xs$ belongs to $B(E, E)^+$ for all s in S. If F is another ordered Banach space, an operator T in ${B(E, F)_S}^+$ will be called positively and invariantly nice if $T^* y$ is an extreme point ${B(E^*)_S}^+$ for each y in ext $B(F^*)^+$.

THEOREM 3. *Let* E *be a separable* C_0 *space and* F *a simplex space. Let a left amenable topological semigroup* S *act on* E *positively from the right. If either* (1) *or* (2) *are satisfied, then* $T \in {B(E, F)_S}^+$ *is extreme there if and only if it is positively and invariantly nice.*

Proof. In this case, the projection $P : E \to E$ which appeared in the proof of Theorem 2 is positive. Therefore PE (with the induced order) is a C_0 space [23, p. 162]. The result now follows by [9, Theorem 10] (= [10, Theorem 13]).

In the setting of Theorem 3, assume that $E = C(X)$ and $F = C(Y)$. Then PE is a C (K) space [23, p. 160]. Denote the units in E , F and PE by 1 and identify Y with its image in F^*. Suppose, in addition, that $1s = 1$ for all s in S. $A = \{T \in B(E, F)_S : T1 = 1\}$ (which is not empty) can be identified with $\{T \in B(PE, F) : T1 = 1\}$. It follows (see, for example, [8, p. 343]) that $T \in A$ is extreme there if and only if $T^* y$ is extreme in $\{z \in B(E^*)_S^+ : \|z\| = 1\}$ for each y in Y. Thus we have obtained a different proof of the equivalence $(a) \Longleftrightarrow (b)$ in [2, Corollary 4.6]. (Here there is no need to assume that E is separable).

4. EXTREMELY AMENABLE SEMIGROUPS

In this section we consider a certain class of discrete semigroups. These are the extremely amenable semigroups [5, p. 46] which were introduced by Mitchell and extensively studied by Granirer. Recall that a discrete semigroup is called extremely left amenable if $LUC(S) = C(S)$ admits a multiplicative left invariant mean. If a discrete semigroup acts on a Banach space E from the right, then it acts on E^* from the left: $\langle x, sy \rangle = \langle xs, y \rangle$, where $x \in E$, $y \in E^*$, and $s \in S$. Information concerning Choquet Theory can be found in [22] and [1].

THEOREM 4. *Let a discrete extremely left amenable semigroup.* S *act on a Lindenstrauss space* E, *and assume that* $E^*_S \neq \{0\}$. *Denote* $B(E^*)$ *by* K *and equip it with the weak star topology. Suppose that* $sy \in \text{ext}\, K$ *for each* y *in* ext K, *and that* ext K *carries every maximal measure on* K. *Then* $\text{ext}\, K_S = K_S \cap \text{ext}\, K$.

Proof. Let $M'_+(K)$ denote the set of all probability measures on K. S acts on C(K) from the right: $(fs)(k) = f(sk)$ where $f \in C(K)$, $s \in S$ and $k \in K$. Therefore we can say that S acts on $M'_+(K)$ from the left. Let y belong to ext K_S. Then $\|y\| = 1$. Lazar's theorem [14] implies [7, p. 444] that there is a unique maximal measure in $M'_+(K)$ which represents y. Denote it by w and let $x \in E$. Since $\langle x, sw \rangle = \langle xs, w \rangle = \langle xs, y \rangle = \langle x, sy \rangle$. sw represents $sy = y$. But sw is carried by ext K. Thus sw is maximal and $sw = w$ for all s in S. Suppose $w = \frac{1}{2}(m_1 + m_2)$ where $m_i \in M'_+(K)_S$. Let y_i denote the resultants of m_i. Then $\langle xs, y_i \rangle = \langle xs, m_i \rangle = \langle x, m_i \rangle = \langle x, y_i \rangle$ for all $x \in E$ and $s \in S$. Thus $y_i \in K_S$. But $y = \frac{1}{2}(y_1 + y_2)$ and $y \in \text{ext}\, K_S$. Hence $y_1 = y_2 = y$. Also, the m_i are maximal. It follows that w is extreme in $M'_+(K)_S$. It must be multiplicative on C(K) [11, p. 58]. Hence w is a point measure and $y \in \text{ext}\, K$.

This result was inspired by [21, p. 244]. We conjecture that it remains true when the restriction " ext K carries every maximal measure on K " is dropped.

COROLLARY. *Let* $E = C_\Sigma(Q)$ *be a separable* C_Σ *space,* F *a* C_σ *space, and* S *a discrete extremely left amenable semigroup (under composition) of continuous self-mapping of* Q *which commute with* Σ. *If we define an action* *on* E *by* $xs(q) = x(sq)$, *where* $x \in \mathrm{E}$, $s \in \mathrm{S}$ *and* $q \in \mathrm{Q}$, *then* $\mathrm{T} \in \mathrm{B}(\mathrm{E}, \mathrm{F})_{\mathrm{S}}$ *is extreme there if and only if it is invariantly nice.*

Proof. Note that $\mathrm{E}^*{}_{\mathrm{S}} \neq \{0\}$ in this case. (This follows from Mitchell's fixed point theorem [17, p. 196]). $\mathrm{E}^*{}_{\mathrm{S}}$ is a conjugate L-space and ext $\mathrm{B}(\mathrm{E}^*)_{\mathrm{S}}$ is weak star closed (Theorem 4 can be applied because $\mathrm{B}(\mathrm{E}^*)$ with the weak star topology is metrizable). Therefore $\mathrm{B}(\mathrm{E}, \mathrm{F})_{\mathrm{S}}$ can be identified with B (G, F) for some separable C_Σ space G [16, p. 336]. The result now follows by [9, Theorem 11].

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