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Oscillations for forced second order nonlinear differential equations

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Equazioni differenziali ordinarie non lineari. — *Oscillations for forced second order nonlinear differential equations.* Nota [(*)] di VASILIOS A. STAIKOS e YIANNIS G. SFICAS, presentata dal Socio G. SANSONE.

RIASSUNTO. — Si danno risultati sul comportamento asintotico ed oscillatorio delle soluzioni di un'equazione differenziale perturbata della forma

$$[l(t)\,\psi(x)\,x']' + a(t)\,\varphi(x) = b(t)$$

o di forma più generale

$$[l(t)\,\psi(x)\,x']' + a(t)\,g(x, x') = b(t)$$

senza la restrizione $a \geq 0$. Questi risultati generalizzano e migliorano altri precedenti dovuti a Bobisud [2] e Kartsatos [3] che riguardano il caso speciale $\psi \equiv 1$, $l \equiv 1$, $b \equiv 0$.

In this paper we are concerned with the oscillatory and asymptotic behavior of solutions for differential equations of the form

$$(*) \qquad [l(t)\,\psi(x)\,x']' + a(t)\,\varphi(x) = b(t)$$

where the nonnegativity of the function a is not assumed. As references to the subject we mention here the papers by Bhatia [1], Bobisud [2] and Kiguradze [4] in which the special case $\psi \equiv 1$ and $b \equiv 0$ is treated. Using the results obtained here for differential equations of the form (*), we derive, by comparison, other ones on oscillatory and asymptotic behavior of solutions for more general differential equations, namely for differential equations of the form

$$(**) \qquad [l(t)\,\psi(x)\,x']' + f(t)\,g(x, x') = b(t)$$

All functions in (*) and (**) are supposed to be real-valued with domains

$$\mathfrak{D}(l) = \mathfrak{D}(a) = \mathfrak{D}(b) = \mathfrak{D}(f) = [t_0, \infty)$$
$$\mathfrak{D}(\psi) = \mathfrak{D}(\varphi) = \mathbf{R} \text{ and } \mathfrak{D}(g) = \mathbf{R}^2.$$

Moreover, ψ is nonnegative, l positive and such that

$$(1) \qquad \int_{t_0}^{\infty} \frac{dt}{l(t)} = \infty .$$

Throughout the sequel, by " solution " of (*) or (**) we shall mean only solutions which are defined for all large t. Also, we shall consider the oscillatory character of the solutions in the usual sense, i.e, a solution is called *oscillatory* if and only if it has no last zero, otherwise it is called *nonoscillatory*.

(*) Pervenuta all'Accademia il 20 agosto 1973.

THEOREM 1. *Consider the differential equation* (*) *subject to the following conditions*:

(i) *the function* φ *is differentiable on* $\mathbf{R} - \{0\}$ *and for every* $x \neq 0$,

$$x\,\varphi(x) > 0 \quad \textit{and} \quad \varphi'(x) \geqq 0$$

(ii) *the function* ψ/φ *is locally integrable on* $(0, \infty)$ *and* $(-\infty, 0)$ *and such that*

$$\int^{\infty} \frac{\psi(x)}{\varphi(x)}\,dx < \infty \quad \textit{and} \quad \int^{-\infty} \frac{\psi(x)}{\varphi(x)}\,dx < \infty$$

(iii) *for every* $\mu > 0$,

$$\int_{t_0}^{\infty} [a(t) - \mu\,|\,b(t)\,|] \int_{t_0}^{t} \frac{ds}{l(s)}\,dt = \infty .$$

Then, every solution x *of* (*) *is oscillatory or such that*

$$\liminf_{t\to\infty} |\,x(t)\,| = 0 .$$

Proof. Let x be a nonoscillatory solution of (*) with $c \equiv \liminf_{t\to\infty} |\,x(t)\,| \neq 0$.

Without loss of generality we can suppose that the domain of x is the whole half-line $[t_0, \infty)$ and that

$$|\,x(t)\,| > \frac{c}{2} \qquad \text{for every} \quad t \geqq t_0 .$$

Moreover, this solution can be supposed positive, since the substitution $u = -x$ transforms (*) into an equation of the same form satisfying the assumptions of the theorem.

For

$$z(t) = -\frac{l(t)\,\psi[x(t)]\,x'(t)}{\varphi[x(t)]} \int_{t_0}^{t} \frac{ds}{l(s)} \tag{2}$$

we have

$$z'(t) = -\frac{[l(t)\,\psi[x(t)]\,x'(t)]'\,\varphi[x(t)] - l(t)\,\psi[x(t)]\,[x'(t)]^2\,\varphi'[x(t)]}{\varphi^2[x(t)]} \int_{t_0}^{t} \frac{ds}{l(s)} -$$

$$-\frac{l(t)\,\psi[x(t)]\,x'(t)}{\varphi[x(t)]}\,\frac{1}{l(t)} \geqq -\frac{[l(t)\,\psi[x(t)]\,x'(t)]'}{\varphi[x(t)]} \int_{t_0}^{t} \frac{ds}{l(s)} - \frac{\psi[x(t)]}{\varphi[x(t)]}\,x'(t) .$$

Since

$$-\frac{[l(t)\,\psi[x(t)]\,x'(t)]'}{\varphi[x(t)]} = a(t) - \frac{b(t)}{\varphi[x(t)]} \geqq a(t) - \frac{|\,b(t)\,|}{\varphi[x(t)]} \geqq$$

$$\geqq a(t) - \frac{1}{\varphi\left(\frac{c}{2}\right)}\,|\,b(t)\,|$$

we obtain that for every $t \geqq t_0$

$$z'(t) \geqq [a(t) - \mu \mid b(t) \mid] \int_{t_0}^{t} \frac{ds}{l(s)} - \frac{\psi[x(t)]}{\varphi[x(t)]} x'(t)$$

where $\mu = \dfrac{1}{\varphi\left(\dfrac{c}{2}\right)}$. This inequality, by integration, gives

$$z(t) \geqq z(t_0) + \int_{t_0}^{t} [a(u) - \mu \mid b(u) \mid] \int_{t_0}^{u} \frac{ds}{l(s)} du - \int_{x(t_0)}^{x(t)} \frac{\psi(x)}{\varphi(x)} dx \geqq$$

$$\geqq z(t_0) + \int_{t_0}^{t} [a(u) - \mu \mid b(u) \mid] \int_{t_0}^{u} \frac{ds}{l(s)} du - \int_{x(t_0)}^{\infty} \frac{\psi(x)}{\varphi(x)} dx$$

from which, by (ii) and (iii), it follows that for some $t_1 > t_0$

$$z(t) \geqq 1 \quad \text{for every} \quad t \geqq t_1 .$$

Thus, by (2), for every $t \geqq t_1$

$$\frac{\psi[x(t)]}{\varphi[x(t)]} x'(t) = - \frac{z(t)}{l(t) \displaystyle\int_{t_0}^{t} \frac{ds}{l(s)}} \leqq - \frac{1}{l(t) \displaystyle\int_{t_0}^{t} \frac{ds}{l(s)}} \tag{3}$$

from which it is obvious that x' is negative on $[t_1, \infty)$ and consequently $\lim_{t \to \infty} x(t) = c$. Integrating, now, the inequality (3) from t_1 to t, we obtain that

$$\int_{x(t_1)}^{x(t)} \frac{\psi(x)}{\varphi(x)} dx \leqq - \int_{t_1}^{t} \frac{du}{l(u) \displaystyle\int_{t_0}^{u} \frac{ds}{l(s)}} = - \left[\log \int_{t_0}^{u} \frac{ds}{l(s)} \right]_{t_1}^{t} . \tag{4}$$

Hence, by (1),

$$\int_{c}^{x(t_1)} \frac{\psi(x)}{\varphi(x)} dx = \infty \tag{5}$$

which contradicts (ii).

Remark. In particular for $\psi \equiv 1$, a function φ satisfying (i) and (ii) can be defined by

$$\varphi(x) = x^{\alpha}$$

where $\alpha > 1$, $\alpha = p/q$ and p, q are odd integers.

THEOREM 2. *Consider the differential equation* (*) *with* $b \equiv 0$, *i.e. the equation*

$$[l(t) \psi(x) x']' + a(t) \varphi(x) = 0 .$$

Then, under the assumptions of Theorem 1, every solution of the differential equation under consideration is either oscillatory or tending monotonically to zero as $t \to \infty$. Moreover, under the additional assumption

$$\text{(iv)} \qquad \int_{0+} \frac{\psi(x)}{\varphi(x)}\, dx < \infty \quad \textit{and} \quad \int_{0-} \frac{\psi(x)}{\varphi(x)}\, dx < \infty$$

all solutions are oscillatory.

Proof. Let x be a nonoscillatory solution. This solution can be supposed again with domain $[t_0, \infty)$ and positive. Using the transformation (2) and following the same technique as in the proof of Theorem 1, we derive at first the inequality

$$z'(t) \geqq a(t) \int_{t_0}^{t} \frac{ds}{l(s)} - \frac{\psi[x(t)]}{\varphi[x(t)]}\, x'(t) \quad , \quad t \geqq t_0 .$$

Then, by integration, we obtain (3) and consequently (4). Since ,by (3), x' is negative on $[t_1, \infty)$, it follows that $\lim_{t\to\infty} x(t) = c \geqq 0$ exists. Thus (4) gives (5), which contradicts (ii) for $c > 0$ and (iv) for $c = 0$.

Remark. The above theorem has been proved in the particular case $\psi \equiv 1$ and $l \equiv 1$ by Bobisud ([2] Theorems 1 and 2) under the additional condition

$$0 < \int_{t_0}^{\infty} a(t)\, dt \leqq \infty .$$

THEOREM 3. *Consider the differential equation* (**) *and the function* φ *subject to the conditions* (i), (ii) *and the following ones*:

(v) *the function g is continuous and such that for every $x \neq 0$ and y,*

$$xg(x, y) > 0$$

(vi) *for every $\mu_1 > 0$ and $\mu_2 > 0$,*

$$\int_{t_0}^{\infty} [\mu_1 f^+(t) - f^-(t) - \mu_2 \,|\, b(t)\,|] \int_{t_0}^{t} \frac{ds}{l(s)}\, dt = \infty$$

where

$$f^+(t) = \max\{f(t), 0\} \ \textit{and} \ f^-(t) = \max\{-f(t), 0\},$$

(vii) $\mathfrak{F}$ *is a class of functions defined and differentiable for all large t, which possesses the property*:

for every $x \in \mathfrak{F}$ with $\liminf_{t\to\infty} |\, x(t)\,| \neq 0$, there exist positive constants L, M *(depending on x) such that*

$$\text{(6)} \qquad L \leqq \frac{g[x(t), x'(t)]}{\varphi[x(t)]} \leqq M$$

for all large t.

Then every solution x of (**) *which belongs to the function class* $\mathfrak{F}$ *is oscillatory or such that*

$$\liminf_{t\to\infty} |x(t)| = 0 .$$

Proof. Let x be a nonoscillatory solution of (**) with $x \in \mathfrak{F}$ and $\liminf_{t\to\infty} |x(t)| \neq 0$. This solution can be supposed with domain $[t_0, \infty)$, positive and such that (6) is satisfied for all $t \geqq t_0$.

If

$$a(t) = f(t) \frac{g[x(t), x'(t)]}{\varphi[x(t)]}$$

then the differential equation (*) has x as a solution. Since, by (6),

$$\begin{aligned} a(t) - \mu |b(t)| &= f(t) \frac{g[x(t), x'(t)]}{\varphi[x(t)]} - \mu |b(t)| \\ &= f^{+}(t) \frac{g[x(t), x'(t)]}{\varphi[x(t)]} - f^{-}(t) \frac{g[x(t), x'(t)]}{\varphi[x(t)]} - \mu |b(t)| \\ &\geqq \mathrm{L} f^{+}(t) - \mathrm{M} f^{-}(t) - \mu |b(t)| \\ &= \mathrm{M}\,[\mu_1 f^{+}(t) - f^{-}(t) - \mu_2 |b(t)|] \end{aligned}$$

where $\mu_1 = \mathrm{L}/\mathrm{M}$ and $\mu_2 = \mu/\mathrm{M}$, we have that (iii) follows from (vi). Thus, by Theorem 1, x must be oscillatory or such that $\liminf_{t\to\infty} |x(t)| = 0$, a contradiction.

Remark. If in the above theorem $\mathfrak{F}$ is the class of all bounded and differentiable functions for all large t, then, as conclusion, we have that *every bounded solution x of* (**) *is oscillatory or such that* $\liminf_{t\to\infty} |x(t)| = 0$. Thus a result due to Kartsatos ([3] Theorem 1) follows from Theorem 3 in the particular case $\psi \equiv 1$, $l \equiv 1$ and $b \equiv 0$.

Applying the same technique used in the above theorem, it is obvious that we can obtain, by Theorem 2, the following theorems.

THEOREM 4. *Consider the function φ and the differential equation* (**) *with $b \equiv 0$, i.e. the equation*

$$[l(t)\,\psi(x)\,x']' + f(t)\,g(x, x') = 0 .$$

Then, under the conditions (i), (ii), (v), (vi) *and*

(vii)′ $\mathfrak{F}$ *is a class of functions defined and differentiable for all large t, which possesses the property*:

for every nonoscillatory $x \in \mathfrak{F}$ with $\lim_{t\to\infty} x(t) \neq 0$, *there exist positive constants* L, M *such that* (6) *is satisfied for all large t*

every solution of the differential equation under consideration, which belongs to the function class $\mathfrak{F}$ is oscillatory or tending to zero as $t \to \infty$.

THEOREM 5. *Consider the function* φ *and the differential equation* (**) *with* $b \equiv 0$. *Then, under the conditions* (i), (ii), (iv), (v), (vi) *and*

(vii)″ $\mathfrak{F}$ *is a class of functions defined and differentiable for all large* t, *which possesses the property*:

for every nonoscillatory $x \in \mathfrak{F}$, *there exist positive constants* L, M *such that* (6) *is satisfied for all large* t

all solutions of the differential equation under consideration, which belong to the function class $\mathfrak{F}$ *are oscillatory.*

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