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A boundedness theorem for some non-linear differential equations of the fourth order

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Equazioni differenziali ordinarie. — *A boundedness theorem for some non-linear differential equations of the fourth order.* Nota (*) di H. O. TEJUMOLA, presentata dal Socio G. SANSONE.

RIASSUNTO. — L'Autore, con opportune ipotesi, dimostra un teorema sulla limitatezza delle soluzioni dell'equazione

$$x^{(iv)} + \varphi(\ddot{x})\dddot{x} + \psi(\dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = \theta(t, x \dot{x}, \ddot{x}, \dddot{x}) .$$

1. This Note is sequel to an earlier one [2] in which the equation

$$(1.1) \qquad x^{(iv)} + \varphi_1(\ddot{x})\dddot{x} + \varphi_2(\ddot{x}) + \varphi_3(\dot{x}) + a_4 x = p(t, x, \dot{x}, \ddot{x}, \dddot{x})$$

was studied, subject to certain generalized Routh-Hurwitz conditions on φ_1, φ_2 and φ_3. Our interest here is in the boundedness result ([2]; Theorem 2) which we extend to the much more general equations of the form

$$(1.2) \qquad x^{(iv)} + \varphi(\ddot{x})\dddot{x} + \psi(\dot{x}, \ddot{x}) + g(x, \dot{x}) + h(x) = \theta(t, x, \dot{x}, \ddot{x}, \dddot{x}) .$$

The functions ψ, g which correspond to φ_2, φ_3 in (1.1) now depend also on $\dot{x}$ and x respectively and the function h, corresponding to $a_4 x$ in (1.1), is non-linear.

Assume here that the functions φ, ψ, g, h, θ are continuous and that $h'(x)$, $\frac{\partial g}{\partial x}(x, y)$, $\frac{\partial \psi}{\partial y}(y, z)$ exist and are continuous for all x, y and z. Let $g_x(x, y) = \frac{\partial g}{\partial x}(x, y)$ and let g_y, ψ_y be similarly defined.

THEOREM. *Let*

$$(1.3) \qquad \psi(y, 0) = g(x, 0) = 0, \; h(x) \operatorname{sgn} x \to +\infty \quad \textit{as} \; |x| \to \infty$$

and suppose that a_i $(i = 1, 2, 3, 4)$, δ_1, δ_2, Δ_0, Δ_1, Δ_2, A *are positive constants such that*

$$(1.4) \qquad \varphi(z) \geq a_1 \;\; \textit{for all } z, \; g(x, y)/y \geq a_3 \;\; \textit{for all } x \;\; \textit{and } y \neq 0$$

$$\textit{and } 0 \leq a_4 - h'(x) \leq \varepsilon \Delta_2 a_1^2, \;\; \textit{for all } x \quad , \quad \Delta_2 = a_2(a_1 + a_3 a_4^{-1});$$

$$(1.5) \qquad \{a_1 a_2 - g_y(x, y)\} a_3 - a_1 \varphi(z) h'(x) \geq \Delta_0 \;\; \textit{for all } x, y \;\; \textit{and } z;$$

$$(1.6) \qquad g_y(x, y) - \frac{g(x, y)}{y} \leq \delta_1 \;\; \textit{for } |y| \geq \Delta_1 \quad , \quad \delta_1 < 2 a_4 \Delta_0/(a_1 a_3^2);$$

$$(1.7) \qquad \frac{1}{z}\int_0^z \varphi(s)\, ds - \varphi(z) \leq \delta_2 \;\; \textit{for } z \neq 0 \quad , \quad \delta_2 < 2 \Delta_0/(a_1 a_3^2);$$

(*) Pervenuta all'Accademia il 27 luglio 1973.

(1.8) $$0 \leq \frac{\psi(y, z)}{z} - a_2 \leq \varepsilon_0 \frac{a_3^3}{a_4^2} \quad \text{for } z \neq 0 ;$$

(1.9) $$g_x^2(x, y) \leq (\varepsilon - \varepsilon_0)\, a_1 \Delta_0 \quad , \quad \frac{1}{y}\int_0^y g_x(x, s)\, ds \leq \frac{(\varepsilon - \varepsilon_0)\, a_3}{4} ,$$

$$\left| \frac{1}{z}\int_0^z \psi_y(y, s)\, ds \right| \leq \frac{\Delta_0}{4\, a_3}$$

for all x , y *and* z;

(1.10) $$|\theta(t, x, y, z, u)| \leq A \quad \text{for all } t, x, y, z \text{ and } u ;$$

(1.11) $$\varepsilon_0 < \varepsilon = \min\left[\frac{a_4}{a_3}, \frac{1}{a_1}, \frac{\Delta_0}{16\, a_1 a_3 \Delta_2}, \frac{a_1}{4\Delta_2}\left(\frac{2\Delta_0}{a_1^2 a_3^2} - \delta_2\right), \right.$$
$$\left. \frac{a_3}{4\, a_4 \Delta_2}\left(\frac{2\, a_4 \Delta_0}{a_1 a_3^2} - \delta_1\right)\right] .$$

Then there exists a constant M_0 *which depends only on* A, φ, ψ, *and* h *such that every solution* $x(t)$ *of* (1.1) *satisfies*

$$|x(t)| \leq M_0 \quad , \quad |\dot{x}(t)| \leq M_0 \quad , \quad |\ddot{x}(t)| \leq M_0 \quad , \quad |\dddot{x}(t)| \leq M_0$$

for all sufficiently large t.

The condition (1.5) is a refinement of the corresponding one (1.4) of [2] which holds for all $y \neq 0$.

2. Let $V = V(x, y, z, u)$ be the continuous function defined by

(2.1) $$V = V_0 + V_1$$

where

(2.2) $$2V_0 = 2d_2\int_0^x h(s)\, ds + (a_2 d_2 - a_4 d_1)\, y^2 + 2\int_0^y g(x, s)\, ds$$
$$+ 2\int_0^z \{s\varphi(s) - d_2 s\}\, ds + d_1 u^2 + 2 d_1 \int_0^z \psi(y, s)\, ds + 2yh(x)$$
$$+ 2 d_1 zh(x) + 2zu + 2 d_1 zg(x, y) + 2 d_2 yu + 2 d_2 y \int_0^z \varphi(s)\, ds$$
$$d_1 = \varepsilon + a_1^{-1} \quad , \quad d_2 = \varepsilon + a_4 a_3^{-1} ,$$

and

(2.3) $$V_1 = \begin{cases} (u + \Phi)\operatorname{sgn} x , & |x| \geq |u + \Phi| \\ x \operatorname{sgn}(u + \Phi) , & |x| \leq |u + \Phi| \end{cases} , \quad \Phi(z) = \int_0^z \varphi(s)\, ds .$$

For reasons which were outlined in [2], the theorem will follow once it is shown that

$$\text{(2.4)} \qquad V(x, y, z, u) \to +\infty \quad \text{as} \quad x^2 + y^2 + z^2 + u^2 \to \infty$$

and that the limit

$$\dot{V}^+ = \limsup_{h \to +0} [V(x(t+h), y(t+h), z(t+h), u(t+h)) - V(x(t), y(t), z(t), u(t))]/h$$

exists, corresponding to any solution $(x(t), y(t), z(t), u(t))$ of the equivalent system

$$\text{(2.5)} \qquad \dot{x} = y \ , \quad \dot{y} = z \ , \quad \dot{z} = u \ , \quad \dot{u} = -\varphi(z)\, u - \psi(y, z), -g(x, y) - h(x) + \theta(t, x, y, z, u)$$

(obtained from (1.2) by setting $\dot{x} = y, \dot{y} = z, \dot{z} = u$), and satisfies

$$\text{(2.6)} \qquad \dot{V}^+ \leq -1 \quad \text{provided} \quad x^2(t) + y^2(t) + z^2(t) + u^2(t) \geq M_1 ,$$

for some constant $M_1 > 0$.

3. First we verify (2.4). Assume that all the conditions of the theorem hold. Then, as in [1], the functions φ, g satisfy

$$\text{(3.1)} \qquad d_1 - 1/\varphi(z) \geq \varepsilon \ \text{for all } z \ , \quad d_2 - a_4\, y/g(x, y) \geq \varepsilon \ \text{for } y \neq 0$$
$$\text{and } a_2 - d_1 g_y(x, y) - d_2 \varphi(z) \geq \frac{\Delta_0}{a_1 a_3} - \Delta_2 \varepsilon \ \text{for all } y, z .$$

Let γ be the function defined by

$$\text{(3.2)} \qquad \gamma(x, y) = \begin{cases} g\dfrac{(x, y)}{y} , & y \neq 0 \\ g_y(x, 0), & y = 0 . \end{cases}$$

Let $z \neq 0$. Then, in analogy with ([1]; pp. 141), V_0 can be rearranged as follows:

$$\text{(3.3)} \qquad 2V_0 = \left\{2 d_2 \int_0^x h(s)\, ds - h^2(x)/\gamma\right\} + \{a_2 d_2 - a_4 d_1 - d_2^2\, \Phi(z)/z\}\, y^2$$
$$+ \left\{2 \int_0^y g(x, s)\, ds - y g(x, y)\right\} + \left\{2 d_1 \int_0^z \psi(y, s)\, ds - d_2 - d_1^2 \gamma\right\} z^2$$
$$+ \left\{2 \int_0^z s\varphi(s)\, ds - z\Phi(z)\right\} + \{d_1 - z/\Phi(z)\}\, u^2$$
$$+ z/\Phi(z)\, [u + \Phi(z) + d_2 y \Phi(z)/z]^2 + \frac{1}{\gamma} [h(x) + y\gamma + d_1 z\gamma]^2 .$$

The first term in curly brackets clearly satisfies

$$\text{(3.4)} \qquad 2 d_2 \int_0^x h(s)\, ds - \frac{h^2(x)}{\gamma} \geq 2\varepsilon \int_0^x h(s)\, ds - \frac{h^2(0)}{a_3} .$$

As for the next two terms, note that

$$2\int_0^y g(x,s)\,ds - yg(x,y) = \int_0^y \left\{\frac{g(x,s)}{s} - g_y(x,s)\right\} s\,ds\,,\ y \neq 0$$

since

$$yg(x,y) = \int_0^y g(x,s)\,ds + \int_0^y s g_y(x,s)\,ds\,.$$

Furthermore

$$\left| 2\int_0^y g(x,s)\,ds - yg(x,y)\right| \leq 3\,\Delta_1^2\,a_1\,a_2 \qquad \text{if} \quad |y| \leq \Delta_1\,,$$

since $|g(x,y)| \leq a_1 a_2 |y|$ for all x and y, the latter being a consequence of the fact that g satisfies (1.4) and (1.5). Since $\delta_1 < \dfrac{2\,\Delta_0\,a_4}{a_1\,a_3^2}$, the various arguments employed in estimating the term V_1 in ([1]; pp. 141–142) can now be applied to the second and third terms in (3.3). Here, by considering the cases: $|y| \leq \Delta_1$, $|y| \geq \Delta_1$ separately, it will seen that

$$(3.5) \qquad (a_2\,d_2 - a_4\,d_1 - d_2^2\,\Phi(z)/z)\,y^2 + 2\int_0^y g(x,s)\,ds - yg(x,y)$$

$$\geq M_2\,y^2 - M_2\,\Delta_1^2 - 3\,\Delta_1^2\,a_1\,a_2$$

with

$$\varepsilon \leq \frac{1}{4}\,a_3\,[2\,\Delta_0\,a_4/(a_1\,a_3^2) - \delta_1]/(a_4\,\Delta_2)\,,$$

where

$$M_2 = \left(\Delta_0\,a_4/(a_1\,a_3^2) - \frac{1}{2}\,\delta_1\right).$$

The fourth and fifth terms in (3.3) correspond to V_2 ([1]; (3.10)]). Since $\psi(y,z)/z \geq a_2\,(z \neq 0)$ and φ, g satisfy (3.1) the estimate for V_2 in [1] holds here also. That is,

$$(3.6) \qquad \left\{2\,d_1\int_0^z \psi(y,s)\,ds - d_2 - d_1^2\,\gamma\right\} z^2 + 2\int_0^z s\varphi(s)\,ds - z\Phi(z) \geq M_3\,z^2$$

where $M_3 = \frac{1}{2}\,[2\,\Delta_0/a_1^2\,a_3 - \delta_2]$ and $\varepsilon \leq \frac{1}{2}\,M_3\,a_1\,\Delta_2^{-1}$. Lastly,

$$(3.7) \qquad \{d_1 - z/\Phi(z)\}\,u^2 \geq \varepsilon u^2\,,$$

since $\Phi(0) = 0$ implies that $\Phi(z) = z\varphi(\tau z)$, $(\theta \leq \tau \leq 1)$, and φ satisfies (3.1). The estimates (3.4), (3.5), (3.6), (3.7) combined with (3.3) show that in the case $z \neq 0$,

$$(3.8) \qquad 2\,V_0 \geq 2\,\varepsilon\int_0^x h(s)\,ds + M_2\,y^2 + M_3\,z^2 + \varepsilon u^2 - M_4$$

where $M_4 = \frac{h^2(0)}{a_3} + M_2 \Delta_1^2 + 3 \Delta_1^2 a_1 a_2$. In the case $z = 0$, it is a simple matter to show that V_0 an estimate of the form (3.8) holds but with the term $M_3 z^2$ absent.

To complete the verification of (2.4), observe now from (2.1) and (2.3) that

$$(3.9) \qquad 2V \geq 2\varepsilon \int_0^x h(s)\, ds - |x| + M_2 y^2 + M_3 z^2 + \varepsilon u^2 - M_4 ,$$

since $|V_1| \leq |x|$. Now if $x \geq 0$

$$2\varepsilon \int_0^x h(s)\, ds - |x| = 2\varepsilon \int_0^x \left\{ h(s) - \frac{1}{2} \varepsilon^{-1} \right\} ds$$

and the integral on the right tends to $+\infty$ as $|x| \to \infty$ since $h(x) \operatorname{sgn} x \to +\infty$ as $|x| \to \infty$. The case $x \leq 0$ can be handled similarly. Therefore

$$2\varepsilon \int_0^x h(s)\, ds - |x| \to +\infty \qquad \text{as} \quad |x| \to \infty,$$

and, in view of (3.9), (2.4) holds.

4. Next we verify (2.6). Let $(x, y, z, u) \equiv (x(t), y(t), z(t), u(t))$ be any solution of (2.5). The existence of $\dot{V}^+ = \dot{V}_0 + \dot{V}_1^+$ is not in question. As for the actual value of $\dot{V}^+$, a simple calculation will show that

$$(4.1) \qquad \dot{V}_0 = -U_1 y^2 - U_2 z^2 - U_3 u^2 - U_4 + d_1 \{h'(x) - a_4\} yz + $$
$$+ d_1 g_x(x, y) yz + (d_2 y + z + d_1 u)\, \theta(t, x, y, z, u)$$

where

$$U_1 = \left[d_2 \frac{g(x, y)}{y} - h'(x) - \frac{1}{y} \int_0^y g_x(x, s)\, ds \right],$$

$$U_2 = \left[a_2 - d_1 g_y(x, y) - \Phi(z)/z - \frac{d_1}{z} \int_0^z \psi_y(y, s)\, ds \right],$$

$$U_3 = d_1 \varphi(z) - 1 ,$$

$$U_4 = z\psi(y, z) - a_2 z^2 + d_2 y (\psi(y, z) - a_2 z) ,$$

and that

$$(4.2) \qquad \dot{V}_1^+ = \begin{cases} -h(x) \operatorname{sgn} x - (\psi(y, z) + g(x, y) + \theta) \operatorname{sgn} x , & |x| \geq |u + \Phi| \\ y \operatorname{sgn}(u + \Phi) , & |x| \leq |u + \Phi| \end{cases} .$$

Here both the co-efficients U_4 and U_3 can be estimated as in the corresponding case (3.5) of [2]. In fact the estimates there show that

$$U_4 \geq -\varepsilon_0 a_3 y^2 \quad \text{for all } y, z \quad ; \quad U_3 \geq \varepsilon a_1 \quad \text{for all } z .$$

The term U_1 satisfies

$$U_1 \geq \varepsilon \frac{g(x,y)}{y} + (a_4 - h'(x)) - \frac{1}{y} \int_0^y g_x(x,s)\, ds$$

since $d_2 = \varepsilon + a_4 a_3^{-1}$ and $\frac{g(x,y)}{y} \geq a_3 \ (y \neq 0)$. Hence

$$U_1 y^2 + U_4 \geq \left\{ \varepsilon' a_3 + (a_4 - h'(x)) - \frac{1}{y} \int_0^y g_x(x,s)\, ds \right\} y^2, \quad \varepsilon' = \varepsilon - \varepsilon_0 .$$

Note now that

$$(a_4 - h'(x))(y^2 - d_1 yz) \geq -(a_4 - h'(x)) \frac{d_1^2}{4} z^2$$

$$\frac{\varepsilon'}{4} a_3 y^2 + d_1 g_x(x,y) yz \geq -\frac{d_1^2 z^2}{\varepsilon' a_3} g_x^2(x,y)$$

for all x, y and z. Therefore

$$U_1 y^2 + U_4 + d_1 (a_4 - h'(x)) yz + d_1 g_x(x,y) yz$$

$$\geq \frac{\varepsilon'}{4} a_3 y^2 - (a_4 - h'(x)) \frac{d_1^2}{4} z^2 - \frac{d_1^2}{\varepsilon' a_3} g_x^2(x,y) z^2$$

since $\frac{1}{y} \int_0^y g_x(x,s)\, ds \leq \frac{\varepsilon' a_3}{4}$, $\varepsilon' = \varepsilon - \varepsilon_0$. Lastly the estimates for U_2 in ([1]; pp. 144) show clearly that

$$U_2 \geq \left(\Delta_0 / a_1 a_3 - \Delta_2 \varepsilon - \frac{d_1}{z} \int_0^z \psi_y(y,s)\, ds \right),$$

so that, in view of (1.4) and (1.9),

$$\left\{ U_2 - (a_4 - h'(x)) \frac{d_1^2}{4} - \frac{d_1^2}{\varepsilon' a_3} g_x^2(x,y) \right\} > \Delta_0 / (8\, a_1 a_3)$$

with $\varepsilon < \Delta_0 / (16\, a_1 a_3 \Delta_2)$. The various estimates obtained for U_i, $i = 1, 2, 3, 4$ together with (4.1) imply that $\dot{V}_0$ satisfies

$$\dot{V}_0 \leq -M_5 (y^2 + z^2 + u^2) + M_6 (|y| + |z| + |u|) \tag{4.3}$$

for some constants $M_5 > 0$, $M_6 > 0$.

The expression for $\dot{V}_1^+$ in (4.2) yields

$$\dot{V}_1^+ \leq \begin{cases} -h(x) \operatorname{sgn} x + M_7 (1 + |y| + |z|), & |x| \geq |u + \Phi| \\ |y| & , |x| \leq |u + \Phi| \end{cases}$$

for some constant $M_7 > 0$, since ψ and θ satisfy (1.8) and (1.10) respectively. Thus, in view of (4.3), there are constants $M_8 > 0$, $M_9 > 0$ such that $\dot{V}^+$ satisfies

$$\dot{V}^+ \leq -M_5 (y^2 + z^2 + u^2) - h(x) \operatorname{sgn} x + M_7 (1 + |y| + |z| + |u|) \tag{4.4}$$

or

$$\dot{V}^+ \leq - M_5 (y^2 + z^2 + u^2) + M_8 (|y| + |z| + |u|) \tag{4.5}$$

according as $|x| \geq |u + \Phi|$ or $|x| \leq |u + \Phi|$.

The usual two-stage argument will now yield (2.6). Indeed since h is continuous and satisfies (1.3), there is a constant $M_9 > 0$ such that $h(x) \operatorname{sgn} x \geq - M_9$ for all x. Thus, it will be clear from either of the two estimates (4.4) and (4.5) that $\dot{V}^+$ satisfies

$$\dot{V}^+ \leq - 1 \qquad \text{provided} \quad y^2 + z^2 + u^2 \geq M_{10}, \tag{4.6}$$

for some constants $M_{10} > 0$. Suppose, however, that $y^2 + z^2 + u^2 \leq M_{10}$ and let $M_{11} = \max_{|u|, |z| \leq M_{10}} |u + \Phi|$. Then $|x| \geq M_{11}$ implies $|x| > |u + \Phi|$ and so, in this case, the estimate (4.4) is applicable. Since $y^2 + z^2 + u^2 \leq M_{10}$, (4.4) implies the existence of a constant $M_{12} \geq M_{11}$ such that

$$\dot{V}^+ \leq - 1 \quad \text{if} \quad y^2 + z^2 + u^2 \leq M_{10} \quad \text{but} \quad |x| \geq M_{12}. \tag{4.7}$$

The two results (4.6) and (4.7) together verify (2.6).

Remarks. The extension of the present result to an equation (1.2) with θ satisfying

$$|\theta(t, x, y, z, u)| \leq A + B (y^2 + z^2 + u^2)^{1/2}, \qquad A > 0, B > 0$$

can be carried out as in [2]. In fact, all that is necessary is to increase each of the estimates (4.4) and (4.5) with a term not exceeding $BM_{13} (y^2 + z^2 + u^2)^{1/2}$, $M_{13} = \max(1, d_1, d_2)$. However if the constant B is fixed so that $B < M_5/(2 M_{13})$ the whole of the estimates for $\dot{V}^+$ will remain true.

References

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[2] H. O. Tẹjumọla, « Atti Accad. Naz. Lincei, Rend. Sci. fis. mat. e nat. », ser. VIII, *52* (1), 16–23 (1972).