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Invertible convolutions

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Calcolo operazionale. — *Invertible convolutions*. Nota di EDGAR BERZ, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Sia D′ lo spazio delle distribuzioni in R^n, dotato della topologia di Schwartz e sia L (D′) lo spazio degli operatori lineari continui $D' \to D'$. In L (D′) gli operatori che sono commutabili con tutte le traslazioni formano una sottoalgebra A (D′) che è isomorfa con l'algebra di convoluzione delle distribuzioni finite. Usando questo isomorfismo e un teorema di Paley–Wiener–Schwartz si prova che gli operatori $A \in A(D')$, che sono invertibili, sono unicamente le traslazioni e multipli non nulli di esse.

As is well known, the finite distributions on R^n form an algebra E′ with respect to the convolution-product; Dirac's measure δ is the unit of this algebra. We propose to determine the invertible elements in this algebra.

The algebra E′ (∗) is isomorphic to the algebra A (D′) of the continuous linear operators $A : D' \to D'$ of the distribution-space D′, which commute with all translations. Therefore the knowledge of the invertible $S \in E'$ leads to the invertible operators $A \in A(D')$. It turns out that these are exactly the translations and the non-zero multiples of them.

1. BASIC CONCEPTS

Let

$$E = C^{\infty}(R) \quad , \quad D = C_0^{\infty}(R).$$

A sequence in E tends to zero in the sense of Schwartz, if it converges to zero uniformly on every compact set and if the same is true for every derivative of this sequence. A sequence in D tends to zero in the sense of Schwartz, if it does so as a sequence of E and if all its functions are concentrated on a fixed compact set.

The spaces D′, E′.

A linear form T on D is a distribution, if

$$\lim T(\varphi_i) = 0$$

for every Schwartz-sequence $\{\varphi_i\}$. The linear space of all distributions shall be denoted by D′

$S \in D'$ is *finite*, if supp S is compact. Every finite $S \in D'$ can be extended to a linear form $\tilde{S}$ on E in such a way, that

$$\tilde{S}(\chi_i) \to 0 \tag{1}$$

(*) Nella seduta del 19 giugno 1973.

for every Schwartz-sequence $\{\chi_i\}$ in E. This extension is unique and it is given by

$$\tilde{S}(\chi) = S(\alpha\chi),$$

where α is a (fixed) test function, which equals 1 on a neighbourhood of supp S.

On the other hand, every linearform S_1 on E, which is continuous in the sense of (1), is the extension $\tilde{S}$ of a finite $S \in D'$.

The space of all finite $S \in D'$ shall therefore be denoted by E'.

Convolutions.

For S, $T \in E'$ we define the convolution $S * T$ by

$$(S*T)(\varphi) = S_{t_1}(T_{t_2}(\varphi(t_1 + t_2))),$$

where $\varphi \in D$. $S * T$ is again finite, we have

$$\text{supp } S*T \subseteq \text{supp } S + \text{supp } T.$$

The extension of $S*T$ to R is given by

$$\widetilde{S*T}(\chi) = \tilde{S}_{t_1}(\tilde{T}_{t_2}(\chi(t_1 + t_2))). \tag{2}$$

In the sequel we will drop $\sim$.

As is well known, the convolution $*$ makes E' into a commutative algebra with unit δ, where

$$\delta(\varphi) = \varphi(0).$$

Fourier-Transformation.

For $S \in E'$ the Fourier-transform $\hat{S} = \psi$ is defined by

$$\psi(s) = S_t(e^{-ist}) \qquad \text{for} \quad s \in C.$$

By virtue of the continuity of S ψ is an entire function. The " Fourier-Transformation ".

$$F : S \to \hat{S}$$

has the following properties:

i) F is linear,

ii) $F(S*T) = F(S)\,F(T)$.

ii) is checked easily by means of (2).

In addition, the following " Inversion-Formula " holds:

iii) $\langle S , \varphi \rangle = \langle \hat{S} , \hat{\varphi} \rangle$,

where

$$\langle S , \varphi \rangle = S(\varphi) \quad , \quad \hat{\varphi} = \frac{1}{2\pi} \int_R e^{ist}\, \varphi(t)\, dt,$$

$$\langle \hat{S} , \hat{\varphi} \rangle = \int_R \hat{S}\hat{\varphi}\, ds.$$

In particular iii) shows that F is injective. If we introduce the linear space $Z = F(E')$, F becomes a linear isomorphism of E', Z.

The functions $\psi \in Z$ can be characterized by the

Theorem of Paley–Wiener–Schwartz.

An entire function $\psi(s)$ is the Fourier-transform of some $S \in E'$, if and only if an estimation

$$(3) \qquad |\psi(s)| \leqq C(1 + |s|)^p e^{a|\tau|} \quad , \quad s = \sigma + i\tau ,$$

holds, where C, a are positive constants, p a nonnegative integer.

For later application we give a simple example:

For $h \in R$ we define the distribution δ_h by

$$\delta_h(\varphi) = \varphi(h).$$

Clearly supp $\delta_h = \{h\}$. The Fourier-transform of δ_h is given by

$$\delta_h(e^{-ist}) = e^{-ish} .$$

2. Invertible Convolutions

Under F the convolution-algebra E' is isomorphic to the algebra Z with its natural operations. In particular this implies that E' has no divisors of zero, a statement of the Titchmarsh type. Of course, we could use this fact to imbed E' into its quotient-field, following the lines of Mikusinski.

From this point of view the question arises which elements S of the algebra E' are invertible in the sense that there exists a $T \in E'$ such that

$$(4) \qquad S * T = \delta .$$

Let us assume in the sequel that S is invertible, and derive necessary conditions for S.

(4) implies for the Fourier-transforms $\hat{S}$, $\hat{T}$:

$$\hat{S}\hat{T} = 1 .$$

Thus the entire function $\psi = \hat{S}$ has no zeros in C. It therefore can be represented in the form

$$(5) \qquad \psi(s) = e^{h(s)},$$

where $h(s)$ is an entire function.

On the other hand, ψ satisfies the estimation (3) of Paley-Wiener-Schwartz. In this inequality the right hand side can be enlarged by

$$e^{c|s|+d},$$

where c, d are positive constants, sufficiently large. We then obtain the condition

$$|e^{h(s)}| \leqq e^{c|s|+d},$$

which is equivalent to

(6) $$\operatorname{Re} h(s) \leqq c\,|s| + d.$$

Information about h itself is available from

Caratheodory's Inequality.

Let $f(z)$ be an entire function, satisfying $f(0) = 0$, $R > 0$,

$$M(R) = \operatorname{Max}\{\operatorname{Re} f(z) : |z| = R\}.$$

Then for every $z \in C$ with $|z| = r < R$ we have

$$|f(z)| \leqq M(R)\,\frac{2r}{R-r}.$$

(See Titchmarsh [6], page 174).

If for a given z we take $R = 2r$, we obtain

(7) $$|f(z)| \leqq 2M(2r).$$

We apply (7) to the function h, assuming, that $h(0) = 0$: in view of (6) we have

$$M(R) \leqq cR + d,$$

hence by (7):

$$|h(s)| \leqq 4c\,|s| + 2d.$$

Thus h is of linear growth at most, hence by Liouville's Theorem h is linear, say

(8) $$h(s) = As + B,$$

where A , B are complex constants.

If $h(0) \neq 0$, we conclude that $h - h(0)$ is linear and therefore h itself.

There remains the question which constants A , B may really occur in (8).

By (3) we have the condition that

$$|e^{As+B}| \leqq C(1 + |s|)^p\, e^{a|\tau|},$$

for $s = \sigma + i\tau$. In particular, if we put $\tau = 0$, we must have

$$|e^{A\sigma+B}| \leqq C(1+|\sigma|)^p.$$

Assuming $A = \alpha + i\beta$ this is equivalent to

$$e^{\alpha\sigma}\,|e^{B}| \leqq C(1 + |\sigma|)^p \qquad \text{for } \sigma \in R.$$

But this estimation can hold only, if $\alpha = 0$.

The conclusion is that h is of the form

$$h(s) = i\beta s + B,$$

where β is a real constant, B a complex one. Hence ψ itself has the form

$$\psi(s) = ce^{i\beta s}, \qquad \text{where } c \in \mathrm{C} - \{0\}.$$

By the example, given in 1), ψ is nothing else than the Fourier-transform of

$$\mathrm{S} = c\delta_h, \qquad \text{where } h = -\beta.$$

Thus every invertible $\mathrm{S} \in \mathrm{E}'$ is of this form. Conversely, every such S is invertible, with

$$\mathrm{S}^{-1} = c^{-1}\,\delta_{-h}.$$

The result is the following

THEOREM 1. *A distribution* $\mathrm{S} \in \mathrm{E}'$ *is invertible if and only if*

$$\mathrm{S} = c\delta_h,$$

where $h \in \mathrm{R}$, $c \in \mathrm{C} - \{0\}$.

In the last proof it was sufficient to know that the Fourier-transform ψ of S has no zeros in C. We therefore have the following

COROLLARY 1. *A distribution* $\mathrm{S} \in \mathrm{E}'$ *is invertible if and only if its Fourier-transform* $\hat{\mathrm{S}}$ *has no zeros in* C.

For the space Z itself we have the following

COROLLARY 2. *All functions* $\psi \in \mathrm{Z}$ *have zeros in* C *except the functions of the form*

$$ce^{i\beta s} \quad , \quad c \in \mathrm{C} - \{0\} \quad , \quad \beta \in \mathrm{R}.$$

3. GENERALIZATION TO SEVERAL VARIABLES

The last Theorem may be generalized to the convolution–algebra $\mathrm{E}'(\mathrm{R}^n)$, consisting of the finite distributions S on R^n, $n \geqq 2$.

For such a distribution the Fourier-transform $\psi = \hat{\mathrm{S}}$ is defined by

$$\psi(s) = \mathrm{S}(e^{-ist}),$$

where $s = (s_1, \cdots, s_n) \in \mathrm{C}^n$, $t = (t_1, \cdots, t_n) \in \mathrm{R}^n$,

$$st = \sum_1^n s_i t_i.$$

$\psi(s)$ is an entire function in $s_1, \cdots, s_n$.

The Fourier-transformation $\mathrm{F}: \mathrm{S} \to \hat{\mathrm{S}}$ is an algebra-isomorphism of $\mathrm{E}'(*)$ and $\mathrm{Z}_n = \mathrm{F}(\mathrm{E}'(\mathrm{R}^n))$. According to the Paley-Wiener-Schwartz Theorem in its general form, the functions $\psi \in \mathrm{Z}_n$ are characterized by the validity of an estimation

(9) $$|\psi(s)| \leqq \mathrm{C}(1 + |s|)^p e^{a|\tau|}$$

where

$$|s| = \left(\sum_1^n |s_i|^2\right)^{1/2} \quad , \quad |\tau| = \left(\sum_1^n \tau_i^2\right)^{1/2} .$$

THEOREM 2. *A distribution* $S \in E'(R^n)$ *is invertible, if and only if*

$$S = c\delta_h ,$$

where $c \in C - \{0\}$, $h = (h_1 , \cdots , h_n) \in R^n$.

Proof. Let $S \in E'(R^n)$ be invertible. Then $\psi = \hat{S}$ has no zeros in C^n, hence

$$\psi(s) = e^{h(s)} , \tag{10}$$

where h is an entire function in $s = (s_1 , \cdots , s_n)$.

We consider the function ψ_1 in the complex variable z, defined by

$$\psi_1(z) = \psi(z , \cdots , z) .$$

By virtue of (9) ψ_1 belongs to Z and by Corollary 2 ψ_1 is of the form

$$\psi_1(z) = e^{\gamma z + \delta} ,$$

where γ , δ are constants. Together with (10) it follows that $h(s)$ is linear, say

$$h(s) = \sum_{j=1}^n A_j s_j + B .$$

Since the function

$$s_1 \to \psi(s_1 , 0 , \cdots , 0) = e^{A_1 s_1 + B}$$

belongs to Z, we conclude from the case $n = 1$, that

$$A_1 = i\beta_1 , \quad \beta_1 \in R .$$

Applying this argument to every s_j, we obtain

$$\psi(s) = c e^{i\left(\sum_1^n \beta_j s_j\right)} , \quad \beta_j \in R , \quad c = e^B .$$

Therefore S itself has the form

$$S = c\delta_h \quad , \quad h = (h_1 , \cdots , h_n) \quad , \quad h_i = -\beta_i ,$$

which proves our assertion.

4. THE ALGEBRA A(D)

Let L(D) be the linear space of the linear operators

$$A : D \to D,$$

which are continuous in the sense that $A\varphi_i \to 0$ for every Schwartz-sequence $\{\varphi_i\}$. With respect to the composition-product $A_1 A_2$ the space L(D) is a (noncommutative) algebra.

Especially every translation-operator τ_h,

$$(\tau_h \varphi)(t) = \varphi(t+h),$$

$h \in \mathrm{R}$, belongs to L(D).

Next we consider the subalgebra A(D) of all operators $\mathrm{A} \in \mathrm{L}(\mathrm{D})$, which commute with the translations, i.e.

$$\tau_h \mathrm{A} = \mathrm{A} \tau_h$$

for all $h \in \mathrm{R}$. They correspond to the distributions $\mathrm{S} \in \mathrm{E}'$ in the sense of the following proposition.

PROPOSITION 1. *Every* $\mathrm{A} \in \mathrm{A}(\mathrm{D})$ *has a unique representation*

$$(11) \qquad (\mathrm{A}\varphi)(s) = \mathrm{S}_t(\varphi(s+t))$$

by a distribution $\mathrm{S} \in \mathrm{E}'$. *Conversely* (11) *defines an operator* $\mathrm{A} \in \mathrm{A}(\mathrm{D})$ *for every* $\mathrm{S} \in \mathrm{E}'$.

Clearly S is unique, since for $s = 0$ we must have

$$\mathrm{S}(\varphi) = (\mathrm{A}\varphi)(0).$$

Conversely it can be shown that, by this equation, a distribution $\mathrm{S} \in \mathrm{E}'$ is given which represents A in the sense of (11).

Let us now consider the mapping σ: $\mathrm{A} \rightarrow \mathrm{S}$, which associates with $\mathrm{A} \in \mathrm{A}(\mathrm{D})$ the representing $\mathrm{S} \in \mathrm{E}'$. Clearly σ is a linear isomorphism. In addition, by a simple computation it can be shown that

$$\sigma(\mathrm{A}_1 \mathrm{A}_2) = \mathrm{S}_1 * \mathrm{S}_2,$$

that means σ is an algebra-isomorphism. Thus we have

PROPOSITION 2. *The algebras* A(D) *and* E′ *are isomorphic under* σ.

Hence A(D) is commutative and has no divisors of zero. The isomorphism σ leads also to

THEOREM 3. *The invertible operators* $\mathrm{A} \in \mathrm{A}(\mathrm{D})$ *are exactly the operators*

$$\mathrm{A} = c\tau_h,$$

$h \in \mathrm{R}$, $c \in \mathrm{C} - \{0\}$.

Indeed, these operators A correspond to the distributions $c\delta_h$.

5. THE ALGEBRA A(D′)

Let L(D′) denote the linear space of the linear operators

$$\mathrm{B} : \mathrm{D}' \rightarrow \mathrm{D}',$$

which are continuous in the sense that $\mathrm{BT}_i \rightarrow 0$ for every sequence of distributions $\mathrm{T}_i \in \mathrm{D}'$, converging to zero.

Every $\mathrm{A} \in \mathrm{L}(\mathrm{D})$ generates an operator $\mathrm{B} \in \mathrm{L}(\mathrm{D}')$ as its transposed $\mathrm{B} = \mathrm{A}'$, i.e. by the formula

$$\langle \mathrm{BT}, \varphi \rangle = \langle \mathrm{T}, \mathrm{A}\varphi \rangle.$$

Conversely, from the reflexivity of D it follows that every $B \in L(D')$ is the transposed of a certain $A \in L(D)$.

The mapping $k : A \rightarrow A'$ therefore is a linear isomorphism of $L(D)$ and $L(D')$. In addition we have

$$(A_1 A_2)' = A_2' A_1'.$$

Especially to $\tau_h \in L(D)$ there corresponds the "translation-operator" τ_h', which for a continuous function $f \in D'$ gives the ordinary translation

$$(\tau_h' f)(t) = f(t-h).$$

Finally we study the operators $B \in L(D')$, which commute with all τ_h', i.e.

$$\tau_h' B = B \tau_h' \qquad \text{for} \quad h \in R.$$

Equivalent is the condition that $A = k^{-1} B$ satisfies

$$A \tau_h = \tau_h A.$$

The operators B in question therefore form the class $kA(D)$, which will be denoted by $A(D')$. Since $A(D)$ is commutative, the same is true for $A(D')$. The induced mapping

$$k : A(D) \rightarrow A(D')$$

is therefore an algebra isomorphism. So we have

PROPOSITION 3. *The algebras* $A(D'), A(D), E', Z$ *are isomorphic.*

From the isomorphism $k : A(D) \rightarrow A(D')$ and Theorem 3 we deduce

THEOREM 4. *The invertible operators* $B \in A(D')$ *are exactly the operators*

$$B = c \tau_h',$$

where $c \in C - \{0\}, h \in R$.

Remark. In view of 3) all considerations in 4) generalize to the case of several variables.

REFERENCES

[1] BERZ E., *Verallgemeinerte Funktionen und Operatoren*. Mannheim, BI–Verlag (1967).
[2] BERZ E., *Operatoren Verallgemeinerter Funktionen*, «Mathematische Annalen», *158*, 215–232 (1965).
[3] GELFAND I. M. and SHILOV G. E., *Generalized Functions*. New York, London, Academic Press (1968).
[4] MIKUSINSKI J., *Operational Calculus*. London, New York, Pergamon Press (1959).
[5] SCHWARTZ L., *Théorie des distributions*. Paris, Hermann (1966).
[6] TITCHMARSH E., *The Theory of Functions*. Oxford, University Press.