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Meaning and truth in the Peano arithmetic

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Logica matematica. — *Meaning and truth in the Peano arithmetic* (*). Nota di ROBERTO MAGARI, presentata (**) dal Socio B. SEGRE.

RIASSUNTO. — Si enunciano alcuni risultati sul tema indicato nel titolo, la dimostrazione dei quali verrà data in un prossimo più ampio lavoro.

Let F be the (first order) Peano arithmetic and let P be the set of the sentences of F.

Starting from some " philosophical " considerations we give a definition of " p is meaningful " (for $p \in \mathrm{P}$) and a new definition of truth according to which:

p *is meaningful iff* p *is true or* $\neg p$ *is true.*

We write $|p$ for " p is meaningful ", $\models p$ for " p is true in the classical (Tarskian) sense (in the standard model) " and $\Vdash p$ for " p is true in the new sense " (i.e. $\models p$ and $|p$).

Let T be the set of the (sentences which are) theorems of F, $\dot{\mathrm{T}}(x)$ a formula which one free variable x which canonically numerates $\mathrm{T}, \mathrm{V} = \{p \in \mathrm{P} : \models p\}$, $\mathrm{V}_0 = \{p \in \mathrm{P} : \Vdash p\}$, $\mathrm{P}_0 = \{p \in \mathrm{P} : |p\}$; we have:

1. V_0 *is the minimum* V' *for which*:
   $\mathrm{T} \subseteq \mathrm{V}'$,
   if $p, p \rightarrow q \in \mathrm{V}'$ then $q \in \mathrm{V}'$,
   if $p \rightarrow \dot{\mathrm{T}}(\bar{p}) \in \mathrm{V}'$ and $p \notin \mathrm{T}$ then $\neg p \in \mathrm{V}'$

(where $\bar{p}$ is the numeral corresponding to the Gödel number for p).

2. $\mathrm{V}_0 \subseteq \mathrm{V}$.

3. *There are two formulas* $\dot{\mathrm{P}}_0(x), \dot{\mathrm{V}}_0(x)$ (*with one free variable*) *such that*:

(i) the following statements are equivalent:
$(i, 1) \models \dot{\mathrm{P}}_0(\bar{p})$
$(i, 2) \Vdash \dot{\mathrm{P}}_0(\bar{p})$
$(i, 3)\ |p$,

(ii) the following statements are equivalent:
$(ii, 1) \models \dot{\mathrm{V}}_0(\bar{p})$
$(ii, 2) \Vdash \dot{\mathrm{V}}_0(\bar{p})$
$(ii, 3) \Vdash p$,

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(iii) for every $p \in P$ we have:

$(iii, 1) \Vdash \dot{T}(\bar{p}) \rightarrow \dot{V}_0(\bar{p})$

$(iii, 2) \Vdash \dot{V}_0(\bar{p}) \wedge \dot{V}_0(\overline{p \rightarrow q}) \rightarrow \dot{V}_0(\bar{q})$

$(iii, 3) \Vdash \dot{V}_0(p \rightarrow \dot{T}(\bar{p})) \wedge \neg \dot{T}(\bar{p}) \rightarrow \dot{V}_0(\overline{\neg p})$

$(iii, 4) \Vdash \dot{P}_0(\bar{p}) \longleftrightarrow \dot{V}_0(\bar{p}) \vee \dot{V}_0(\overline{\neg p})$

(it is possible to strengthen (iii) so that, for example, we have: $\Vdash \forall x(\dot{T}(x) \rightarrow \rightarrow \dot{V}_0(x))$, etc.).

4. P_0, V_0 are in Σ_2 in the arithmetical hierarchy.

We claim that in this way we get a system which in a reasonable sense avoids the usual expressive and deductive limitations (1).

We shall give the proofs of these and other results and an explanation of our last claim in a paper soon to be published.

(1) This is not in contradiction with Gödel's and Tarski's limitative results, but for having such a system we pay, obviously, a price: we suitably enlarge the concept of deduction and restrict the set of meaningful proposition (usually one assumes implicitly that every arithmetical proposition is meaningful). With these agreements the claim is technically almost obvious; the main problem is to decide whether the agreements are in some sense acceptable.