Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Francesco S. De Blasi

On the im plicit function theorem in metric spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.6, p. 877–881.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_6_877_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1973.

Equazioni funzionali. — *On the implicit function theorem in metric spaces.* Nota di FRANCESCO S. DE BLASI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema sulle funzioni implicite negli spazi metrici usando una generalizzazione, dovuta a Browder, del metodo di Picard delle approssimazioni successive. Tale teorema contiene propriamente risultati ottenuti da altri Autori, come risulta da un esempio.

1. The implicit function theorem for equations

$$y = f(x, y) \tag{1}$$

in metric (in particular in Banach) spaces has been proved, under different hypotheses, by many Authors by means of the Picard method of the successive approximations (see [1], [2], [3], [4], [6]). This same method will be used in the present Note as well. Actually we shall adapt for our aim an extension of Picard's method, which is due to Browder [5]. In this way, the existence of a unique continuous function $y(x)$ satisfying (1) is shown (see Theorem 1) under less restrictive conditions on f; moreover an explicit estimate to the deviation of an approximating solution of (1), from the real one, is given. An example shows that Theorem 1 contains properly similar results obtained by other Authors.

2. DEFINITION 1. We denote by χ the class of all non-decreasing functions $\varphi : R^+ \to R^+$, $R^+ = [0, \infty)$, such that: (i) there exists a $\delta > 0$ such that $\lim_{n\to\infty} \varphi^n(\delta) = 0$, where $\varphi^1(\delta) = \varphi(\delta)$, $\varphi^{n+1}(\delta) = \varphi(\varphi^n(\delta))$, $n = 1, 2, \cdots$; (ii) $\varphi(0) = 0$ and $\varphi(r) < r$, if $0 < r \leq \delta$.

It is obvious that, if $\varphi \in \chi$ and for some $\delta > 0$ $\lim_{n\to\infty} \varphi^n(\delta) = 0$ then, for any $0 \leq r < \delta$, we have $\lim_{n\to\infty} \varphi^n(r) = 0$.

We denote by $B_1(x_0, r)$, $B_2(y_0, r)$ closed balls (of radius $r > 0$) in M_1, M_2 (M_i, $i = 1, 2$, a complete metric space).

DEFINITION 2. Let M_1, M_2 be complete metric spaces and denote by d the distance in M_2. The function $f : M_1 \times M_2 \to M_2$ is called a φ-contraction on $B_1(x_0, r_1) \times B_2(y_0, r_2) \subset M_1 \times M_2$ if

$$d(f(x, y_1), f(x, y_2)) \leq \varphi(d(y_1, y_2)),$$

for all $(x, y_1), (x, y_2) \in B_1(x_0, r_1) \times B_2(y_0, r_2)$, where the function φ is in χ and satisfies condition (i) of Definition 1 with $\delta \geq 2r_2$.

(*) Nella seduta del 19 giugno 1973.

THEOREM 1. *Suppose that the continuous function* $f : B_1 \times B_2 \to M_2$, *where* $B_1 = B_1(x_0, k) \subset M_1$, $B_2 = B_2(y_0, k) \subset M_2$ *and* M_1, M_2 *are complete metric spaces, satisfies the hypotheses*:

(i) f *is a* φ*-contraction on* $B_1 \times B_2$;

(ii) $d(f(x_0, y_0), y_0) < k - \varphi(k)$.

Then there exist $0 < h \leq k$ *and a unique continuous function* $y : B \to B_2$, $B = B_1(x_0, h)$, *satisfying*

$$f(x, y(x)) = y(x), \qquad \textit{for all} \quad x \in B. \tag{2}$$

Proof. By hypothesis f is continuous, so there exists $0 < h \leq k$ such that

$$d(f(x, y_0), y_0) < k - \varphi(k), \qquad \text{for all} \quad x \in B = B_1(x_0, h).$$

Define $y_1(x) = f(x, y_0)$, $x \in B$. For all $x \in B$, we have $d(y_1(x), y_0) = d(f(x, y_0), y_0) < k - \varphi(k) \leq k$. Suppose that $y_p(x) = f(x, y_{p-1}(x))$, $x \in B$, satisfies $d(y_p(x), y_0) < k$, for all $x \in B$. Then

$$d(y_{p+1}(x), y_0) \leq d(f(x, y_p(x)), f(x, y_0)) + d(f(x, y_0), y_0) <$$
$$< \varphi(d(y_p(x), y_0)) + k - \varphi(k)$$

and, since φ is non-decreasing,

$$d(y_{p+1}(x), y_0) < \varphi(k) + k - \varphi(k) = k.$$

This shows that the sequence of functions $\{y_p(x)\}$, $x \in B$, is well defined. Clearly all $y_p(x)$ are continuous. Let us prove that $\{y_p(x)\}$ is a Cauchy sequence. For this purpose we shall adapt an argument of Browder [5]. For $n = 2, 3, \cdots$ we have

$$c_n(x) = \sup_{p,q \geq n} d(y_p(x), y_q(x)) = \sup_{p,q \geq n} d(f(x, y_{p-1}(x)), f(x, y_{q-1}(x))) \leq$$
$$\leq \sup_{p,q \geq n-1} \varphi(d(y_p(x), y_q(x))) \leq$$
$$\leq \varphi(\sup_{p,q \geq n-1} d(y_p(x), y_q(x))) = \varphi(c_{n-1}(x)).$$

On the other hand, since for any $x \in B$ $y_p(x)$ and $y_q(x)$ are in B_2, we have

$$c_1(x) = \sup_{p,q \geq 1} d(y_p(x), y_q(x)) \leq 2k.$$

From $c_n(x) \leq \varphi(c_{n-1}(x))$ and the above inequality we obtain

$$c_{n+1}(x) \leq \varphi^n(2k), \qquad x \in B, \qquad n = 1, 2, \cdots.$$

But $\lim_{n \to \infty} \varphi^n(2k) = 0$. So, by virtue of the definition of $c_n(x)$, $\{y_p(x)\}$ is a Cauchy sequence and there exists a unique continuous function $y : B \to B_2$

satisfying

$$\lim_{p\to\infty} y_p(x) = y(x).$$

From $y_{p+1}(x) = f(x, y_p(x))$, letting $p \to \infty$, we obtain $y(x) = f(x, y(x))$, which completes the proof of the existence part of the theorem. Let $y_1 : B \to B_2$ be another continuous function satisfying (2). If, for some $x \in B$, $y(x) \neq y_1(x)$, we have

$$d(y(x), y_1(x)) = d(f(x, y(x)), f(x, y_1(x))) \leq$$
$$\leq \varphi(d(y(x), y_1(x))) < d(y(x), y_1(x)).$$

From the contradiction the uniqueness follows and the proof of the theorem is complete.

COROLLARY 1. *Under the hypotheses of Theorem 1, the deviation* $\rho_p = \sup_{x\in B} d(y_p(x), y(x))$ *of an approximating solution* $y_p(x)$ *of* (2) *from the real one* $y(x)$, *satisfies the inequality*

$$\rho_p \leq \varphi^p(k) \qquad p = 1, 2, \cdots, \tag{3}$$

where $\varphi^p(k) \to 0$, *as* $p \to \infty$.

Proof. Since φ is non-decreasing, we have

$$d(y_p(x), y(x)) \leq \varphi(d(y_{p-1}(x), y(x))) \leq \varphi(\rho_{p-1}).$$

Hence $\rho_p \leq \varphi(\rho_{p-1})$ from which (3) follows, for $\rho_1 \leq \varphi(k)$.

Remark 1. In [5] Browder has proved a fixed point theorem for φ–contractions $f : M_2 \to M_2$ where: (*a*) the function $\varphi : R^+ \to R^+$ is non–decreasing, continuous on the right and such that $\varphi(r) < r$, if $r > 0$. In [6] Santoro has proved Theorem 1 using a particular class of φ–contractions, namely supposing that: (*b*) $\varphi : R^+ \to R^+$, $\varphi(0) = 0$, is a non-decreasing function such that, for some $\delta > 0$, $\sum_{n=1}^{\infty} \varphi^n(\delta)$ converges. It is clear that if φ satisfies (*a*) or (*b*), then $\varphi \in \chi$. If we define $\varphi : R^+ \to R^+$ by $\varphi(x) = 1/(n+1)$, if $x \in (1/(n+1), 1/n]$, $n = 1, 2, \cdots$, $\varphi(0) = 0$, $\varphi(x) = x$, if $x > 1$, then φ is in χ and satisfies neither (*a*) nor (*b*). When in Theorem 1 $\varphi(r) = \alpha r$, $0 \leq \alpha < 1$, we obtain a result due to Dieudonné (see [4], p. 256).

3. In this section we shall assume that M_1, M_2, M_3 are Banach spaces. The norm in any of the spaces M_i $(i = 1, 2, 3)$ will be denoted by $|\cdot|$.

THEOREM 2. *Let* $f : B_1 \times B_2 \to M_3$, *where* $B_1 = B_1(x_0, k) \subset M_1$, $B_2 = B_2(y_0, k) \subset M_2$ *and* M_1, M_2, M_3 *are Banach spaces, be a continuous function. Define* $g : B_1 \times B_2 \to M_2$ *by* $g(x, y) = y + Lf(x, y)$, *where* L *is a linear homeomorphism from* M_3 *onto* M_2. *If*

(i) g *is a* φ*-contraction on* $B_1 \times B_2$,

(ii) $(k - \varphi(k)) |L|^{-1} > |f(x_0, y_0)|$, *then there exist* $0 < h \leq k$ *and a unique continuous function* $y : \mathrm{B} \to \mathrm{B}_2$, $\mathrm{B} = \mathrm{B}_1(x_0, h)$, *satisfying*

$$f(x, y(x)) = 0, \quad \text{for all } x \in \mathrm{B}, \tag{4}$$

where 0 *is the origin in* M_3.

Proof. By hypothesis g is a φ–contraction on $\mathrm{B}_1 \times \mathrm{B}_2$. Moreover

$$|g(x_0, y_0) - y_0| \leq |\mathrm{L}|\, |f(x_0, y_0)| < k - \varphi(k).$$

Then Theorem 1 applies and there exists a unique continuous function $y : \mathrm{B} \to \mathrm{B}_2$ satisfying

$$y(x) + \mathrm{L}f(x, y(x)) = y(x).$$

From this, since L is invertible, (4) follows and the proof of the Theorem is complete.

If we define $y_1(x) = g(x, y_0)$, $y_p(x) = g(x, y_{p-1}(x))$, $x \in \mathrm{B}$, from Corollary 1 and Theorem 2, we have

COROLLARY 2. *Under the hypotheses of Theorem 2, the deviation* $\rho_p = \sup_{x \in \mathrm{B}} d(y_p(x), y(x))$ *of an approximating solution* $y_p(x)$ *of* (4) *from the real one* $y(x)$, *satisfies the inequality* (3), *where* $\varphi^p(k) \to 0$ *as* $p \to \infty$.

COROLLARY 3. *Let* $f : \mathrm{B}_2 \to \mathrm{B}_1$ *be continuous and define* $g : \mathrm{B}_1 \times \mathrm{B}_2 \to \mathrm{M}_2$ *by* $g(x, y) = y + \mathrm{L}(f(y) - x)$, *where* L *is a linear homeomorphism from* M_1 *onto* M_2. *If* g *is a* φ*–contraction on* $\mathrm{B}_1 \times \mathrm{B}_2$ *and if* $(k - \varphi(k))\,|\mathrm{L}|^{-1} > |f(y_0) - x_0|$, *then there exist* $0 < h \leq k$ *and a unique continuous function* $y : \mathrm{B}_1(x_0, h) \to \mathrm{B}_2$ *such that*

$$f(y(x)) = x \quad , \quad x \in \mathrm{B}_1(x_0, h).$$

Remark 2. The above corollary on the local invertibility of a function has been proved, under stronger hypotheses, by Pulvirenti in [3] (see also Nevanlinna [2]).

Example. Denote by $\mathrm{C}(\mathrm{I})$, $\mathrm{I} = [0, 1]$, the Banach space of all continuous functions from I to R and let $\mathrm{B} = \mathrm{B}_1(0, 1) \subset \mathrm{C}(\mathrm{I})$. Suppose that $a : \mathrm{I} \to \mathrm{R}$, $b : \mathrm{I} \times \mathrm{I} \times [-1, 1] \to \mathrm{R}^+$ and $g : [-1, 1] \to \mathrm{R}^+$ are continuous and such that $\|a\| < 1/2$, $\|b\| \leq 1$ and $|g(y) - g(y_1)| \leq \varphi(|y - y_1|)$, for all $|y|, |y_1| \leq 1$, where $\varphi \in \chi$ is such that $\delta \geq 2$. For any fixed $x \in \mathrm{B}$, consider the non-linear integral equation

$$y = f(x, y) \quad \text{where} \quad f(x, y) = a(\cdot) + \int_0^1 b(\cdot, s, x(s))\, g(y(s))\, ds.$$

Clearly f is a continuous φ–contraction on $\mathrm{B} \times \mathrm{B}$. If, in particular, $g(y) = \varphi(|y|)$, where $\varphi(0) = 0$, $\varphi(u) = 1/2$ if $u \geq 1$ and $\varphi(u/(n+1) + (1-u)/n) = u/(n+2) + (1-u)/(n+1)$, $u \in [0, 1]$, $n = 1, 2, \cdots$, and if b is such that $b(t, s, x) = (1 - |x|) c(t, s)$ where $c(0, s) = 1$, then we have $\|f(0, 1/n) - f(0, 0)\| = \varphi(1/n) = 1/(n+1)$ and so neither the theorem of Dieudonné nor that of Santoro applies. However, the hypotheses of Theorem 1 are satisfied (with $x_0 = y_0 = 0$ and $k = 1$). Thus, there exist $0 < h \leq 1$ and a unique continuous function $x \to y(x)$, $x \in B(0, h)$, satisfying $y(x) = f(x, y(x))$, for all $x \in B(0, h)$. In other words, for any $x \in B(0, h)$, the given non-linear integral equation has a unique solution which is a continuous function of x.

REFERENCES

[1] T. H. HILDEBRAND and L. M. GRAVES, *Implicit functions and their differentials in general analysis*, « Trans. Amer. Math. Soc. », *29*, 127–153 (1927).

[2] B. NEVANLINNA, *Über die Methode der sukzessiven Approximationen*, « Ann. Acad. Scient. Fennicae », *A 1*, 291 (1955).

[3] G. PULVIRENTI, *Funzioni implicite negli spazi di Banach*, « Matematiche (Catania) », *16*, 1–7 (1961).

[4] J. DIEUDONNÉ, *Fondements de l'analyse moderne*, Gauthier–Villars, Paris (1965).

[5] F. E. BROWDER, *On the convergence of successive approximations for non-linear functional equations*, « Nederl. Akad. Wetensch. Proc. », Ser. A, *30*, 27–35 (1968).

[6] P. SANTORO, *Alcune generalizzazioni di teoremi esistenziali che utilizzano il procedimento di iterazione*, « Atti Acc. Naz. Lincei, Rend. Cl. Sc. Fis. Mat. Nat. », *46*, 541–544 (1969).