Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Mario Martelli

Some results concerning multi-valued mappings defined in Banach spaces

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.6, p. 865–871.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_6_865_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1973.

Analisi funzionale. — *Some results concerning multi-valued mappings defined in Banach spaces* (*). Nota di MARIO MARTELLI, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra che proiettando su $B = \{x \in X : \|x\| \leq 1$, X spazio di Banach di dimensione non finita$\}$ un insieme compatto e convesso si ottiene un aciclico. Usando tale risultato si dimostrano un Teorema di punto fisso per una classe di applicazioni multivoche non compatte definite su B e un'estensione del Teorema di Birkhoff–Kellogg. Si danno alcune applicazioni di tali risultati.

1. INTRODUCTION

The main purpose of this paper is to prove that a densifying, upper semicontinuous multi-valued mapping $T : B \multimap X$, where B is the unit ball of a Banach space X, has a fixed point if the following two conditions are satisfied:

i) $T(x)$ is convex and closed for every $x \in B$;

ii) $\lambda x \in T(x)$ for some $x \in \partial B$, the boundary of B, implies $\lambda \leq 1$.

I rely heavily on two theorems. The first, proved by L. Vietoris [8], is the following.

THEOREM A. *Let $f : X \to Y$ be a continuous map such that $f(X) = Y$ and $f^{-1}(y)$ is acyclic for every $y \in Y$. If X and Y are compact metric spaces then $f_* : H_*(X) \to H_*(Y)$ is an isomorphism.*

This result has been proved using Vietoris cycles and homologies over a field F of coefficients. It is known that it can be stated in a more general situation when X and Y are not compact and f is proper, provided that we use, for example, the Alexander cohomology with coefficients in an R–module G, where R is a commutative ring with a unit (E. Spanier [1]) or the Vietoris–Čech homology with compact carriers and rational coefficients (A. Granas and J. W. Jaworowski [2]).

The second theorem we will make use of has been proved by S. Eilenberg and D. Montgomery [9] and it says that

THEOREM B. *Let X be a compact, acyclic absolute neighborhood retract and $T : X \multimap X$ be an upper semicontinuous multi–valued map. Assume that $T(x)$ is acyclic for every $x \in X$. Then T has a fixed point.*

To make the understanding of our result easier we would like to note that we had, until now, the following situation.

In 1941 S. Kakutani [4] extended Brouwer's fixed point theorem for the ball B of E^n to the class of upper semicontinuous multi–valued maps with convex and closed values.

(*) Work performed under the auspices of the National Research Council (C.N.R.).
(**) Nella seduta del 19 giugno 1973.

Using convexity arguments H. F. Bohnenblust and S. Karlin [5] proved Schauder's [6] theorem for upper semicontinuous, compact multi–valued maps with convex values.

Meanwhile several improvements were obtained for single–valued maps. B. Knaster, C. Kuratowski and S. Mazurkiewicz [16] proved Brouwer's theorem with the assumption $f(\partial B) \subset B$ instead of the stronger condition $f(B) \subset B$. Later E. Rothe [3] obtained the same result in Banach spaces, generalizing Schauder's theorem.

S. Eilenberg and D. Montgomery [9] proved the result of Knaster-Kuratowski-Mazurkiewicz for the class of upper semicontinuous multi-valued mappings with compact and acyclic values. Later A. Granas [7] gave an analogous extension of Rothe's theorem for the class of upper semicontinuous compact multi-valued maps with convex values.

I tried to weaken the assumptions of compactness of T and the boundary condition $T(\partial B) \subset B$ in Granas' theorem by assuming that T is densifying (see Notations and Definitions) and that $\lambda x \in T(x)$ for some $x \in \partial B$ implies $\lambda \leq 1$ (Theorem 2).

Among the results which are contained in Theorem 2, about to be proved in Section 3, I would like to mention here one of W. V. Petryshyn's [15] theorems, which states that a densifying map $f: B \to X$, where B is the unit ball of a Banach space X, has a fixed point provided that it satisfies the boundary condition $\Pi^{\leq}$ ($\lambda x = f(x)$ for some $x \in \partial B$ implies $\lambda \leq 1$).

In Section 4 I will give a few applications to some surjectivity problems obtaining, as a Corollary, H. Schafers' [10] well-known theorem.

2. Notations and Definitions

Multi-valued maps.

We recall that a multi-valued map T of a set X into a set Y is a triple (G, X, Y) where G, the graph of T, is a subset of $X \times Y$ such that $T(x) = \{y \in Y : (x, y) \in G$ is nonempty for each $x \in X\}$. $T(X) = \cup\{T(x) : : x \in X\}$ is the range of T while X is its domain. I will use the symbol $T : X \multimap Y$ to indicate a multi-valued map and $f: X \to Y$ for the single–valued maps. If $A \subset X$ and $B \subset Y$ then $T(A) = \cup\{T(x) : x \in A\}$, while $T^{-}(B) = \{x \in X : T(x) \cap B \neq \varnothing\}$ and $T^{+}(B) = \{x \in X : T(x) \subset B\}$. $T^{-}(B)$ and $T^{+}(B)$ are called the lower inverse image and the upper inverse image of B respectively. For $f: X \to Y$ we have $f^{-}(B) = f^{+}(B) = f^{-1}(B)$.

Upper semicontinuous multi–valued maps.

Let X and Y be topological spaces and $T : X \multimap Y$. We say that T is upper semicontinuous (u.s.) at $x_0 \in X$ if for any open set O containing $T(x_0)$ there exists a neighborhood $U(x_0)$ such that

a) $x \in U(x_0) \Rightarrow T(x) \subset O$.

If T is upper semicontinuous at each point $x \in X$ then T is said to be u.s. on X. The following two conditions are equivalent to the above.

b) T is u.s. if for any open set $O \subset Y$ the set $T^+(O)$ is open;

c) T is u.s. if for any closed set $C \subset Y$ $T^-(C)$ is closed.

We say that $T : X \multimap Y$ has *closed values* if $T(x)$ is closed for every $x \in X$ and that T has *closed graph* if G, the graph of T, is a closed subset of $X \times Y$. If T has closed graph then it has closed values. If X and Y are metric spaces and $T : X \multimap Y$ has closed graph then $x_n \to x$, $y_n \to y$ and $y_n \in T(x_n)$ implies $y \in T(x)$. It is known that G is closed in $X \times Y$ if T is u.s., Y is regular and T has closed values.

If Y is compact then $T : X \multimap Y$ is u.s. with closed values if and only if T has closed graph.

We say that $T : X \multimap Y$ is compact if for any bounded set $A \subset X$, $T(A)$ is relatively compact. If the stronger condition $T(X) \subset K$, K being a compact subset of Y, is verified, then we say that T is completely continuous.

Fixed points and invariant sets.

A fixed point of a multi-valued map $T : X \multimap X$ is a point $x \in X$ such that $x \in T(x)$. A subset $A \subset X$ is said to be invariant under T if $T(A) \subset A$.

Densifying multi-valued maps.

Let X be a Banach space. For any bounded set $A \subset X$ we define $\alpha(A)$ (C. Kuratowski [11]) as the infimum of all $r > 0$ such that A can be covered by a finite family of subsets with diameter less than r. Let us recall here some properties of this number, called sometimes *measure of noncompactness*.

1) $\alpha(A) = 0 \iff$ A is precompact (= totally bounded);

2) $\alpha(\overline{co}(A)) = \alpha(A)$, where $\overline{co}(A)$ indicates the closed convex hull of A.

A map $T : X \multimap X$ is said to be densifying if $\alpha(T(A)) < \alpha(A)$ for any bounded subset $A \subset X$ such that $\alpha(A) \neq 0$.

Homology.

Let $\mathcal{C}$ be the category of topological spaces, $\mathcal{F}$ be the category of graded vector spaces over a field F. By $H_k(X)$, where $X \in \mathcal{F}$, we denote the k-th Vietoris homology vector space and by $H_*(X)$ the graded vector space associated to X. Given a continuous map $f : X \to Y$ we shall write

$$f_* : H_*(X) \to H_*(Y).$$

A nonempty topological space X is said to be acyclic if $H_i(X) = 0$ for $i \neq 0$ and $H_0(X) \cong F$.

Some other notations.

In what follows, unless otherwise stated, X will be an infinite dimensional Banach space, $B(0, r) = \{x \in X : \|x\| \leq r\}$, $\partial B(0, r) = \{x \in X : \|x\| = r\}$

and $\pi : X \to B(o, r)$ will be the radial retraction of X onto $B(o, r)$. I will say that a mapping $T : B(o, r) —\circ X$ has the property P if $\lambda x \in T(x)$ for some $x \in \partial B(o, r)$ implies $\lambda \leq 1$.

3. Results

In order to prove the first theorem we need the following lemma.

Lemma 1. *Let* $T : B(o, r) —\circ X$ *be densifying. Then for any* $x \in B(o, r)$ *the set* $T(x)$ *is precompact.*

Proof. It cannot be $\alpha(T(x)) \neq o$ because otherwise $\alpha(T(x)) < \alpha(x) = o$.

Theorem 1. *Let* $T : B(o, r) —\circ X$ *be as in Lemma* 1. *Then* $\pi \circ T(x)$ *is acyclic for every* x *such that* $T(x)$ *is closed and convex.*

Proof. $T(x)$ is compact. Since π is continuous $\pi \circ T(x)$ is compact. It is easy to see that $\pi^{-1}(y)$ is acyclic for every $y \in \pi \circ T(x)$. Applying Vietoris' theorem we obtain that

$$\pi_* : H_*(T(x)) \to H_*(\pi \circ T(x))$$

is an isomorphism. Since $T(x)$ is convex we have $H_i(T(x)) = o$ if $i \neq o$ and $H_0(T(x)) \cong F$. Thus $H_i(\pi \circ T(x)) = o$ if $i \neq o$ and $H_0(\pi \circ T(x)) \cong F$.

Lemma 2. *The radial retraction* π *is* α*–nonexpansive.*

Proof. Let A be a bounded set of X. Then $\pi(A) \subset \overline{co}(A \cup \{o\})$. Since $\alpha(\overline{co}(A \cup \{o\})) = \alpha(A)$ it follows that $\alpha(\pi(A)) \leq \alpha(A)$.

The following Lemma 3 has been proved by the Author in a simpler case [13], but the technique used there can be applied also to this one.

Lemma 3. *Let* $T : K —\circ K$ *be a mapping defined in a compact topological space* K. *Then there exists a closed nonempty subset* M *of* K *such that* $M = \overline{T(M)}$.

Note that if T is u.s. with closed values and K is Hausdorff then $T(M)$ is compact, therefore closed. Hence $M = T(M)$.

Theorem 2. *Let* $T : B(o, r) —\circ X$ *be a densifying, u.s. map with convex and closed values. Assume that* T *satisfies condition* P. *Then* F_T, *the set of fixed points of* T, *is nonempty and compact.*

Proof. Since π is α–nonexpansive, $\pi \circ T$ is still densifying.

Moreover $\pi \circ T(x)$ is acyclic for every $x \in B(o, r)$ by Theorem 1.

Since $\pi \circ T$ is densifying the set $K = \overline{\bigcup_n (\pi \circ T)^n (x_0)}$, where $x_0 \in B(o, r)$, is compact. Moreover $\pi \circ T(K) \subset K$. Let M be the subset of K whose existence is insured by Lemma 3 and consider the family

$$\mathfrak{D} = \{D \subset B(o, r) : M \subset D,\ D \text{ closed, convex and invariant under } \pi \circ T\}.$$

Put $C = \cap \{D : D \in \mathfrak{D}\}$. Clearly $\overline{co}\, \pi \circ T(C) = C$. Since $\alpha(\overline{co}(\pi \circ T(C))) = \alpha(\pi \circ T(C)) = \alpha(C)$ it follows that C is compact.

By Theorem B we can find $x \in C$ such that $x \in \pi \circ T(x)$. This implies the existence of a $y \in T(x)$ such that $x = \pi(y)$. If $\|x\| < r$ then $y = x$ because the restriction of π to $B(o, r)$ is the identity and we are done. If $\| x \| = r$ then $y = \lambda x$, with $\lambda \geq 1$. But in this case $\lambda x \in T(x)$ and so, by condition P, $\lambda \leq 1$.

It follows that $\lambda = 1$ i.e. $y = x$. So F_T is nonempty. Clearly $F_T \subset T(F_T)$. This implies $\alpha(F_T) \leq \alpha(T(F_T))$. On the other hand $\alpha(T(F_T)) < \alpha(F_T)$ if $\alpha(F_T) \neq 0$. It follows that $\alpha(F_T) = 0$ and F_T is precompact.

But it is also closed because it is the lower inverse image of the $0 \in X$ under the upper semicontinuous map $I - T$. Indeed $(I - T)^{-}(0) = \{x \in B(o, r) : 0 \in x - T(x) \text{ i.e. } x \in T(x)\} = F_T$.

It follows that F_T is compact.

COROLLARY 1. (A. Granas [7]). *Let* $T : B(o, r) \multimap X$ *be an u.s. map with closed and convex values. Assume that* T *is compact and* $T(x) \subset B(o, r)$ *for every* $x \in \partial B(o, r)$. *Then* T *has a fixed point.*

Remark. Theorem 2 contains, as a particular case, the well-known result of Rothe [3]. It contains also many other theorems which would be too long to mention here. As examples I will give only the following two.

COROLLARY 2. (M. Krasnoselskij [14]). *Let* $f : B(o, r) \to H$ *be a continuous compact map, where* H *is a Hilbert space. If for every* $x \in \partial B(o, r)$.

$$\langle f(x), x \rangle \leq \| x \|^2$$

then f *has a fixed point.*

COROLLARY 3. (W. V. Petryshyn [15]). *Let* $f : B(o, r) \to X$ *be a densifying map which satisfies condition* P. *Then* M, *the set of fixed points of* f, *is nonempty and compact.*

THEOREM 3. (Birkhoff–Kellogg Theorem). *Let* $T : \partial B \multimap X$ *be a compact upper–semicontinuous map with closed and convex values. Assume that* $\inf\{ \| y \| : y \in T(x), x \in \partial B\} \geq \varepsilon > 0$. *Then there exists a point* $x_0 \in \partial B$ *and a real number* $\lambda_0 > 0$ *such that* $x_0 \in \lambda_0 T(x_0)$.

Proof. Define $p : T(\partial B) \to \partial B$, $p(y) = y / \| y \|$. Then the composite map $p \circ T : \partial B \multimap \partial B$ is compact, upper semicontinuous with closed and acyclic values. Since ∂B is an acyclic ANR $p \circ T$ has a fixed point x_0 (L. Gorniewicz and A. Granas [17]). Clearly $x_0 \in \lambda_0 T(x_0)$ for some $\lambda_0 > 0$.

4. APPLICATIONS

The first result of this section is Theorem 4, which is a generalization to multi-valued maps of a theorem obtained by M. Martelli and A. Vignoli (see Corollary 4).

THEOREM 4. *Let* $T : X \multimap X$ *be an u.s. and densifying map with closed and convex values. Assume that there exists a sequence of spheres* $\{\partial B(o, \beta_n)\}$ *and a sequence* $\{\gamma_n\}$ *of positive real numbers* $\gamma_n \to \infty$ *as* $n \to \infty$, *such that for any* $\lambda > 1$ *and any* $x \in \partial B(o, \beta_n)$

$$\inf_{y \in T(x)} \| y - \lambda x \| \geq \gamma_n .$$

Then the equation $y \in x - T(x)$ *has a solution for any* $y \in X$.

Proof. Let $y_0 \in X$ and choose n_0 large enough so that $\| y_0 \| < \gamma_{n_0}$.

Define $T_0(x) = y_0 + T(x)$. Clearly T_0 has the same properties of T. Therefore if $\lambda x \in T_0(x)$ for some $x \in \partial B(0, \beta_{n_0})$ implies $\lambda \leq 1$, Theorem 2 will give the existence of a point $x \in B(0, \beta_{n_0})$ such that $x \in T_0(x)$, i.e. $x \in y_0 + T(x)$, which means $y_0 \in x - T(x)$.

Assume $\lambda > 1$. We have

$$0 = \inf_{z \in T_0(x)} \| z - \lambda x \| = \inf_{y \in T(x)} \| y - y_0 - \lambda x \| \geq \inf_{y \in T(x)} (\| y - \lambda x \| -$$
$$- \| y_0 \|) \geq \gamma_{n_0} - \| y_0 \| > 0 .$$

This contradiction shows that $\lambda \leq 1$ and the theorem is proved.

Remark. With only minor changes we can prove that if $k \geq 1$ then the equation

$$y \in kx - T(x)$$

has a solution for any $y \in X$.

Moreover if we assume that, for any bounded subset A of X,

$$\alpha(T(A)) \leq h\alpha(A) \quad , \quad 0 < h < 1$$

and that the condition

$$\inf_{y \in T(x)} \| y - \lambda x \| \geq \gamma_n$$

holds for any $\lambda > h$ then the equation

$$y \in kx - T(x)$$

has a solution for any $k \geq h$.

COROLLARY 4 (M. Martelli and A. Vignoli [12]). *Let $f: X \to X$ be an α–Lipschitz mapping with constant k and let F be an isomorphism.*

Assume that:

i) $\| F^{-1} \| k \leq 1$

ii) *there exists a sequence of spheres $\partial B(0, \beta_n)$ and a sequence of positive real numbers $\gamma_n \to \infty$ as $n \to \infty$ such that for any $\lambda > 1$ and any $x \in \partial B(0, \beta_n)$*

$$\| f(x) - F(\lambda x) \| \geq \gamma_n .$$

Then the mapping $F - f$ is surjective.

The next Theorem contains, as a Corollary, a well-known theorem of H. Schaefer [10] in a particular case. More precisely the result of Schaefer is valid also in locally convex Hausdorff topological vector spaces, but in this case cannot be obtained as a Corollary of our Theorem.

THEOREM 5. *Let $T: X \multimap X$ be an u.s. and densifying map with convex and closed values. If there is no $x \in X$ such that $x \in T(x)$ then the set $M = \{x \in X : \lambda x \in T(x)$ for some $\lambda > 1\}$ is unbounded.*

Proof. Let $B_n = \{x \in X : \| x \| \leq n\}$ and let π_n be the radial retraction of X onto B_n. Then Theorem 2 gives a point $x_n \in B_n$ such that $x_n \in \pi_n \circ T(x_n)$.

Clearly $\| x_n \| = n$, otherwise we would have $x_n \in T(x_n)$. Moreover there exists $\lambda > 1$ such that $\lambda x_n \in T(x_n)$.

COROLLARY 5. (H. Schaefer [10]). *Let* $f : X \to X$ *be compact and continuous. If there exists* $\lambda_0 \in [0, 1]$ *such that the equation* $x = \lambda_0 f(x)$ *does not have any solutions, then the set* $M = \{x \in X : x = \lambda f(x), 0 < \lambda < \lambda_0\}$ *is unbounded.*

Proof. Suppose that the equation $x = \lambda_0 f(x)$ does not have any solutions. Since f is compact we can apply Theorem 5 to the map $\lambda_0 f$. Therefore for any n we have an element $x_n \in X$ such that $\| x_n \| = n$ and $\lambda_n x_n = \lambda_0 f(x_n)$ with $\lambda_n > 1$. This implies $x_n = \lambda_n^{-1} \lambda_0 f(x_n)$ and $0 < \lambda_n^{-1} \lambda_0 < \lambda_0$.

REFERENCES

[1] E. SPANIER, *Algebraic Topology*, McGraw-Hill, New York, 344 (1966).

[2] A. GRANAS and J. W. JAWOROWSKI, *Some theorems for multivalued mappings of subsets of the Euclidean space*, « Bull. Ac. Pol. Sci., Sér. des sci. math., astr. et phys. », 7 (5), 277–284 (1959).

[3] E. ROTHE, *Zur theorie der topologischen Ordung und der Vectorfelder in Banachschen Räumen*, « Compositio Mathem. », *5*, 177–197 (1937).

[4] S. KAKUTANI, *A generalization of Brouwer's fixed point theorem*, « Duke Math. Journal », *8*, 457–459 (1951).

[5] H. F. BOHNENBLUST and S. KARLIN, *On a theorem of Ville*, Contribution to the theory of games. Vol. 1, 155–160, « Ann. of Math. Studies », *24*, Princeton (1950).

[6] J. SCHAUDER, *Der Fixpunktsatz in Funktionalräum*, « Studia Math. », *2*, 171–180 (1930).

[7] A. GRANAS, *Theorem on antipodes and theorems on fixed points for a certain class of multivalued mappings in Banach spaces*, « Bull. Ac. Pol. Sci., Sér. des sci. math. astr. et phys. », 7 (5), 271–275 (1950).

[8] E. G. BEGLE, *The Vietoris mapping theorem for bicompact spaces*, « Annales of Mathematics », *51*, 534–543 (1950).

[9] S. EILENBERG and D. MONTGOMERY, *Fixed point theorems for multi-valued transformations*, « Am. Journal of Math. », *68*, 214–222 (1946).

[10] H. SCHAEFER, *Über die Methode der a priori Schranken*, « Math. Ann. », *129*, 415–416 (1955).

[11] C. KURATOWSKI, *Topologie*, « Monografie Matematiczne », *20*, Warzawa (1958).

[12] M. MARTELLI and A. VIGNOLI, *Eigenvectors and surjectivity for α-Lipschitz mappings in Banach spaces*, « Ann. di Mat. Pura ed Appl. » (to appear).

[13] M. MARTELLI, *A lemma on maps of a compact topological space and an application to fixed point theory*, « Atti Acc. Naz. Lincei », ser. VIII, *49*, 128–129 (1970).

[14] M. KRASNOSELSKIJ, *New existence theorems for solutions of nonlinear integral equations*, « Dokl. Acad. Nauk. SSSR. », *88*, 949–952 (1953).

[15] W. V. PETRYSHYN, *Structure of the fixed points set of k-set contractions*, « Archiv. Rat. Mech. Anal. », *40* (4), 312–328 (1971).

[16] B. KNASTER, C. KURATOWSKI and S. MAZURKIEWICZ, *Ein Beweis des Fixpunktsatzes für n-dimensionale simplexe*, « Fund. Math. », *14*, 132–137 (1929).

[17] L. GORNIEWICZ and A. GRANAS, *Fixed point theorems for multivalued mappings of the absolute neighborhood retracts*, « J. Math. Pures et Appl. », *49*, 381–395 (1970).