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Fixed points for mappings which are not necessarily continuous

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Topologia. — *Fixed points for mappings which are not necessarily continuous.* Nota di FRANCESCO S. DE BLASI, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra l'esistenza di punti fissi per trasformazioni, non necessariamente continue, di uno spazio metrico completo in sé. Si ottengono, come corollari, noti teoremi dovuti a Edelstein, Browder, Furi e Vignoli.

The aim of this Note is to prove the existence of fixed points for mappings f from a complete metric space into itself, which are not necessarily continuous.

Denote by (M, d) a complete metric space, $(C(M), D)$ the complete metric space of all nonempty closed and bounded subsets of M with the Hausdorff distance $D(A, B) = \max\{\sup_{a\in A} \inf_{b\in B} d(a, b), \sup_{b\in B} \inf_{a\in A} d(a, b)\}$. Observe that $(C(M), D)$ is complete because (M, d) is assumed to be so (see Kuratowski [3], Vol. I, Ch. III). Let $(K(M), D)$ be the subspace of $(C(M), D)$ consisting of all nonempty compact subsets of M. Since this subspace is closed $(K(M), D)$ is complete. Note that (M, d) can be regarded as a closed subspace of both $(C(M), D)$ and $(K(M), D)$ by virtue of the isometry $x \to \{x\}$. For any nonempty subset X of a metric space, $\rho(X)$ will represent the diameter of X. The following definition is due to Kuratowski [3].

DEFINITION 1. Let A be a bounded subset of M. We denote by $\alpha(A)$ the greatest lower bound of all $\varepsilon > 0$ such that A can be decomposed into a finite union of sets of diameter less than ε.

We shall denote by χ the class of all functions $\varphi : R^+ \to R^+$, $R^+ = [0, \infty)$, which are non-decreasing, continuous on the right and which satisfy $\varphi(r) < r$, for all $r > 0$. If a function $f : M \to M$ satisfies $d(f(x), f(y)) \leq \varphi(d(x, y))$, $x, y \in M$, it is called a φ–contraction; if $\varphi(r) = kr$, where $0 \leq k < 1$, f is called a contraction. If f satisfies $d(f(x), f(y)) < d(x, y)$, $x, y \in M$, $x \neq y$, f is called a weak contraction. For any function $g : X \to X$, X any set, if $x_0 \in X$ we define $g^1(x_0) = g(x_0)$, $g^{n+1}(x_0) = g(g^n(x_0))$, $n = 1, 2, \cdots$.

We have:

THEOREM 1. *Suppose that* $f : M \to M$ *satisfies the hypotheses*:

(i) *there exists* $A \in C(M)$ *such that* $f(A) \subset A$;

(ii) *for any* $X \in C(M)$ *such that* $X \subset A$, $f(X) \subset X$ *and* $\rho(X) > 0$, *there exists a proper subset* $Y \in C(M)$ *satisfying* $f(Y) \subset Y \subset X$ *and* $\alpha(Y) \leq \varphi(\alpha(X))$, *where* $\varphi \in \chi$.

(*) Nella seduta del 12 maggio 1973.

Then f has a fixed point. This is unique if f is a contraction, a φ–contraction or a weak contraction.

Proof. If $\rho(A) = 0$, $A = \{a\}$ and a is a fixed point of f. Suppose $\rho(A) > 0$. From (ii) there exists $A_1 \in C(M)$ such that $f(A_1) \subset A_1 \subset A$ and $\alpha(A_1) \leq \varphi(\alpha(A))$. By the same reason, if $\rho(A_1) > 0$ (the case $\rho(A_1) = 0$ is trivial) there exists $A_2 \in C(M)$ such that $f(A_2) \subset A_2 \subset A_1$ and $\alpha(A_2) \leq \leq \varphi(\alpha(A_1)) \leq \varphi^2(\alpha(A))$. So one obtains a sequence $\{A_n\}$, $A_n \in C(M)$, $\rho(A_n) > 0$, which satisfies $f(A_n) \subset A_n \subset A_{n-1}$ and $\alpha(A_n) \leq c_n$, $c_n = = \varphi^n(\alpha(A))$, $n = 2, 3, \cdots$. We have $c_n \to 0$ as $n \to \infty$. In fact $\{c_n\}$ is a non-increasing sequence with terms $c_n \geq 0$ and therefore has limit $\gamma \geq 0$. If $\gamma > 0$, from $c_{n+1} = \varphi(c_n)$ and the continuity of φ on the right, we get $\gamma = \varphi(\gamma)$, from which $\gamma = 0$. Consequently $\alpha(A_n) \to 0$ as $n \to \infty$. Since $A_1 \supset A_2 \supset \cdots$, and $A_n \in C(M)$, $n = 1, 2, \cdots$, by virtue of the Cantor-Kuratowski theorem (see [3], Vol. I, p. 412), $B = \bigcap_{n=1}^{\infty} A_n$ is nonempty and compact, that is $B \in K(M)$. From $B \subset A_n$, $n = 1, 2, \cdots$, we get $f(B) \subset f(A_n) \subset A_n$, hence $f(B) \subset B$. Denote by V the family of all those $X \in K(M)$ satisfying $f(X) \subset X \subset A$. $B \in V$, so V is nonempty. Introduce in V the partial ordering which is induced by the inclusion. Let $(P_s)_{s \in I}$ be any completely ordered subset in V. $P = \bigcap_{s \in I} P_s$ is nonempty and compact so $P \in K(M)$ and, since $f(P) \subset P$, P is in V and is a lower bound for $(P_s)_{s \in I}$. From the Kuratowski-Zorn lemma, there exists in V a minimal element Q. We have $f(Q) \subset Q$. Assume $\rho(Q) > 0$. Then, from (ii), there exists a proper subset $Q_1 \subset Q$, $Q_1 \in K(M)$, such that $f(Q_1) \subset Q_1$, in contradiction to the minimality of Q. Thus $\rho(Q) = 0$ and f has a fixed point. The last statement is obvious.

Remark 1. Similar applications of the Kuratowski-Zorn lemma in theorems of fixed points occur in Kirk [2], Martelli [6] and Jones [7]. (I am indebted to M. Martelli for bringing to my attention the recent work of Jones [7] in which results similar to those of this Note have been proved). Observe that, in the above Theorem, f can be discontinuous.

COROLLARY 1 (Browder [4]). *Let $f: M \to M$, M bounded, be a φ–contraction. Then f has a unique fixed point.*

Proof. By taking $A = M$, hypothesis (i) of the above theorem is satisfied. Let $X \subset A$, $X \in C(M)$, be such that $f(X) \subset X$ and $\rho(X) > 0$. Then $Y = \overline{f(X)}$ is a proper subset of X, because $\rho(Y) \leq \varphi(\rho(X)) < \rho(X)$, and satisfies $f(Y) \subset Y$. By virtue of the definition of α, $\alpha(Y) = \alpha(f(X)) \leq \varphi(\alpha(X))$ and also condition (ii) is fulfilled. Then, by Theorem 1, f has a unique fixed point.

Remark 2. Observe that, under the hypotheses of Corollary 1, for each $x \in M$, $\{f^n(x)\}$ converges to the unique fixed point of f (see Browder [4]).

COROLLARY 2. *Suppose that* $f: M \to M$ *satisfies the hypotheses*:

(i) *there exists* $A \in K(M)$ *such that* $f(A) \subset A$;

(ii) *for any* $X \in K(M)$ *such that* $f(X) \subset X$ *and* $\rho(X) > 0$, *there exists a proper subset* $Y \in K(M)$ *satisfying* $f(Y) \subset Y \subset X$.

Then f has a fixed point.

Proof. By Theorem 1 because, for any $X \in K(M)$, $\alpha(X) = 0$.

Since any weak contraction has at most one fixed point and satisfies condition (ii) of Corollary 2, we have:

COROLLARY 3. *Any weak contraction, in particular a* φ*–contraction or a contraction* $f: M \to M$ *which satisfies hypothesis* (i) *of Corollary* 2, *has a unique fixed point.*

COROLLARY 4 (Edelstein [1]). *Any weak contraction* $f: M \to M$ *such that* $\overline{f(M)} \in K(M)$ *has a unique fixed point.*

Proof. From Corollary 3, because $f(\overline{f(M)}) \subset \overline{f(M)}$.

COROLLARY 5 (Contraction principle). *Any contraction* $f: M \to M$ *has a unique fixed point.*

Proof. Since $\{f^n(x_0)\}$, $x_0 \in M$, is a Cauchy sequence, the set $A = \overline{\bigcup_{n=1}^{\infty} f^n(x_0)}$ is compact, so $A \in K(M)$. Clearly $f(A) \subset A$ and hypothesis (i) of Corollary 2 is satisfied. Then, by Corollary 3, f has a unique fixed point.

COROLLARY 6 (Furi-Vignoli [5]). *Let* $f: M \to M$ *be weakly contractive and such that, for any bounded subset* $X \subset M$, $\alpha(X) > 0$ *implies* $\alpha(f(X)) < \alpha(X)$. *If, for some* $x_0 \in M$, *the sequence* $\{f^n(x_0)\}$ *is bounded, f has a unique fixed point.*

Proof. Following [5], consider the set $A = \bigcup_{n=1}^{\infty} f^n(x_0)$. Clearly $f(A) \subset A$. From $\alpha(A) = \max\{\alpha(f(A)), \alpha(f(x_0))\} = \alpha(f(A))$ and the hypothesis on f, we deduce that $\alpha(A) = 0$. Then $\bar{A} \in K(M)$ and since $f(\bar{A}) \subset \bar{A}$, Corollary 3 furnishes the existence of the fixed point.

Remark 3. Observe that, under the hypotheses of Corollary 6, for any $x \in M$, $\{f^n(x)\}$ converges to the unique fixed point of f (see Furi-Vignoli [5]). Moreover, it is shown in [5] that Corollaries 1, 4 and 5 follow from 6.

DEFINITION 2. Let $f: M \to M$ be given and denote by S a closed subspace of C(M). The function f is called a set-φ contraction on S, if there exists a φ–contraction $F: S \to S$ satisfying $f(X) \subset F(X)$, for any $X \in S$. When $\varphi(r) = kr$, $0 \leq k < 1$, f is called a set-contraction on S.

Since (S, D), S a closed subspace of C(M), is a complete metric space, for any bounded subset $A \subset S$, the number $\alpha(A)$ is well defined.

DEFINITION 3. The function $f: M \to M$ is called a weak set-α contraction on S, S a closed subspace of C(M), if there exists a weak contraction $F: S \to S$ satisfying $f(X) \subset F(X)$, for any $X \in S$, and $\alpha(F(U)) < \alpha(U)$, for any bounded subset $U \subset S$ with $\alpha(U) > 0$.

We have:

THEOREM 2. *Let* $f: M \to M$ *satisfy hypothesis* (ii) *of Corollary* 2. *If any of the following conditions is fulfilled*: (*j*) M *is bounded and* f *is a set-*φ *contraction on* K(M), (*jj*) f *is a set-contraction on* K(M), (*jjj*) f *is a weak set-*α *contraction on* K(M) *and, for some* $X_0 \in K(M)$, $\{F^n(X_0)\}$ *is bounded, then* f *has a fixed point.*

Proof. Assume (*jjj*) and let $F: K(M) \to K(M)$ correspond to f according to Definition 3. Since K(M) is complete and F satisfies the hypotheses of Corollary 6, there exists $A \in K(M)$ such that $A = F(A)$. Therefore $f(A) \subset A$ and the existence of a fixed point for f follows from Corollary 2. The proof is similar when (*j*) (or (*jj*)) holds.

THEOREM 3. *Suppose that* $f: M \to M$ *is a weak set-*α *contraction on* S, S *a closed subspace of* C(M), *and let* $F: S \to S$ *correspond according to Definition 3. If there exist* $X_0, Y_0 \in S$ *such that* $\{F^n(X_0)\}$ *is bounded and* $\lim_{n\to\infty} \rho(F^n(Y_0)) = 0$, *then* f *has a fixed point.*

Proof. By virtue of Corollary 6 and Remark 3, F has a unique fixed point $A = F(A)$, $A \in S$, and we have $F^n(Y_0) \to A$ as $n \to \infty$. Then, since $\rho(F^n(Y_0)) \to 0$ as $n \to \infty$, and ρ is a continuous function on S, we obtain $\rho(A) = 0$. From $f(A) \subset F(A) = A$, we get $\rho(f(A)) = 0$ and f has a fixed point.

Remark 4. By a similar argument one can prove that, if $f: M \to M$ is a set-contraction and the corresponding function F satisfies $\lim_{n\to\infty} \rho(F^n(X_0)) = 0$, for some $X_0 \in S$, then f has a fixed point.

Remark 5. The function $f: R^+ \to R^+$, $f(0) = 0$, $f(x) = x/2 + 1$, $x > 0$, which is not a contraction, is a set-contraction on $K(R^+)$ if one defines F by $F([a, b]) = [0, b/2 + 1]$, $F(X) = F(I(X))$, $X \in K(R)$, I(X) the smallest compact interval containing X. It is clear that the discontinuous function f satisfies the hypotheses of Theorem 2. Moreover $A = [0, 2]$, for $[0, 2] = F([0, 2])$.

Remark 6. Define $f: R^+ \to R^+$ to be any discontinuous function such that $f(x) \in [0, x/2]$, $x \in R^+$. If we take $F([a, b]) = [0, b/2]$, $F(X) = F(I(X))$, $X \in K(R^+)$, then f is a set-contraction on $K(R^+)$. Moreover, for any $X_0 \in K(R^+)$, we have $\rho(F^n(X_0)) \to 0$ as $n \to \infty$ and, by Remark 4, f has a fixed point.

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