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The Cut Locus of a Finsler Manifold

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Topologia. — *The Cut Locus of a Finsler Manifold.* Nota di BADIE T. M. HASSAN, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — In questa Nota si estendono agli spazi di Finsler completi e/o compatti risultati noti negli spazi di Riemann concernenti la totalità delle geodetiche uscenti da un punto.

1. INTRODUCTION

The study of the cut locus (1) of a Riemannian manifold has led to many interesting results in Riemannian geometry. For example, the proof of the so called " Sphere theorem " due to Rauch [6] depends on estimates of the distance to the cut locus. Moreover, it was realized that much of the topological interest of a manifold lies in its cut locus. A very good account of these results and methods are contained in articles by Klingenberg [3], Kobayashi [4], and Weinstein [7].

The aim of this paper is to extend these methods to the study of the cut locus of a Finsler manifold. As in Riemannian geometry, the exponential map is an important tool in forming the proofs. However, for Finsler manifolds this map is not a C^∞ map, as it is only of class C^1 on zero vectors.

2. NOTATIONAL CONVENTIONS

The following notations will be used throughout this paper.

M : a complete connected Finsler manifold of dimension n, $n \geq 2$, endowed with a general metric d. By a general metric we mean one which satisfies all metric properties except the symmetry property.
$T(M)_m$: the tangent space to M at $m \in M$.
$\| X \|$: norm of the tangent vector $X \in T(M)_m$.
exp : the exponential map of $T(M)_m$ onto M.
dexp : the differential of exp.
S : $\{X \mid \|X\| = 1 , X \in T(M)_m\}$.
R^+ : the set of positive real numbers.
γ_X : $\{(t , \gamma_X(t)) \mid \gamma_X(t) = \exp tX , t \in [0 , \infty) , X \in S\}$ is a geodesic starting from m with initial vector X and parametrized by arc-length.
A_X : $\{s \mid$ the segment of γ_X from m to $\gamma_X(s)$ is minimizing, $s \in R^+ \cup \{\infty\}\}$.
$L(\gamma_X)$: the length of γ_X.

(*) Nella seduta del 14 aprile 1973.

(1) For a point m of a manifold M, the cut locus K_m of m in M is the set of all points $p \in M$ such that there exists a minimal segment from m to p which is not minimizing beyond p.

3. The cut locus

From the above definition of the set A_X it follows that:

(1) $s \in A_X \wedge t < s \Rightarrow t \in A_X$,

(2) $r \in R^+ \wedge (s < r \Rightarrow s \in A_X) \Rightarrow r \in A_X$,

(3) $A_X = (0, r]$ for some $r \in R^+ \veebar A_X = R^+ \cup \{\infty\}$.

If $A_X = (0, r]$, then the point $\gamma_X(r)$ is called the cut point of m along γ_X. If $A_X = R^+ \cup \{\infty\}$, then no point of γ_X is a cut point of m.

We define a real valued function

$$c : S \to R^+ \cup \{\infty\}$$

as

$$c(X) = \begin{cases} r & \text{if } A_X = (0, r] \\ \infty & \text{if } A_X = R^+ \cup \{\infty\}. \end{cases}$$

Set $S_0 = c^{-1}(R^+)$. The function

$$f : S_0 \to T(M)_m$$

is defined as $f(X) = c(X) X$. The set $f(S_0) \subset T(M)_m$ is denoted by $\tilde{K}_m$.

The function

$$g : S_0 \to M$$

defined as $g = \exp \circ f$ is such that $g(X)$ is a cut point of m along γ_X. The set $g(S_0) \subset M$ is therefore the set of all cut points of m along all geodesics starting from m. The set $g(S_0)$ is called the cut locus of m in M and is denoted by K_m. It is clear that $\exp \tilde{K}_m = K_m$. The set $\tilde{K}_m$ is called the cut locus of m in $T(M)_m$ and its points are called cut points of m in $T(M)_m$.

From the fact that geodesics do not minimize arc-length beyond the first conjugate point, it follows immediately that

Assertion A. *If p is the first conjugate point of m along γ_X, then there is a point of K_m along γ_X which is not beyond p.*

Assertion B. *If γ_X is a minimal segment from m to p and p is conjugate to m along γ_X, then $p \in K_m$.*

Theorem 3.1. *Let $\{\sigma_i\}$ be a sequence of curves from m to p. If $p \notin K_m$ and limit $L(\sigma_i) = d(m, p)$, then $\{\sigma_i\}$ converges to the unique minimal segment from m to p.*

Proof. Since M is complete, then there exists a minimal segment γ_X from m to p. Set $d(m, p) = b$, $L(\sigma_i) = b_i$, and let

$$\sigma_i = \{(t, \exp tX_i) \mid t \in [0, b_i], X_i \in S\}.$$

For each value of δ, $0 \leq \delta < b$, the set of vectors $(b_i - \delta) X_i$ is contained in some compact subset of $T(M)_m$. We may assume, by taking a subsequence if necessary, that

$$\text{limit } (b_i - \delta) X_i = (b - \delta) Y , \qquad Y \in S .$$

Then,

$$\gamma_Y = \{(t , \exp tY) \mid 0 \leq t \leq b\}$$

is a minimal segment from m to p. It is clear that limit $\sigma_i = \gamma_Y$.

If $X = Y$, then $\gamma_X = \gamma_Y$ and the theorem is proved.

If $X \neq Y$, then $\gamma_X(t)$, $0 \leq t \leq b'$, is no longer minimizing for every b' greater than b. This contradicts the assumption $p \notin K_m$.

Hence the assumption $X \neq Y$ is false and the theorem is proved.

THEOREM 3.2. *If $p \in K_m$ along a geodesic γ_X, then at least one of the following statements holds:*

(1) *p is the first conjugate point of m along γ_X,*

(2) *there exist, at least, two minimizing geodesics from m to p.*

Proof. If $p = \gamma_X(r)$, then we choose a monotone decreasing sequence $\{a_k\}$, $a_k \in R^+$, such that limit $a_k = r$. Let $b_k = d(m , \gamma_X(a_k))$, $k \in N$. Since M is complete, then m and $\gamma_X(a_k)$ can be joined by a minimal segment, namely

$$\sigma_k = \{(t , \exp tX_k) \mid t \in [0 , b_k] , X_k \in S\} .$$

It is clear that

$$X \neq X_k \quad , \quad a_k > b_k \quad , \quad \text{limit } b_k = r .$$

The set of vectors $b_k X_k$ is contained in some compact subset of $T(M)_m$. We may assume, by taking a subsequence if necessary, that

$$\text{limit } b_k X_k = rY \quad , \quad Y \in S .$$

Then,

$$\gamma_Y = \{(t , \exp tY) \mid t \in [0 , r]\}$$

is a minimal segment from m to p.

Now, we have two cases:

Case I. $X = Y$. Then,

$$\exp b_k X_k = \exp a_k X ,$$

and

$$\text{limit } b_k X_k = rX = \text{limit } a_k X ,$$

implie that exp is not one-to-one in a neighborhood U of $rX = rY$. Thus dexp is singular there and p is conjugate to m along γ_X.

On the other hand, if $\gamma_X(s)$, $0 < s < r$, were conjugate to m along γ_X, then γ_X would not be minimizing beyond $\gamma_X(s)$. Hence $p \notin K_m$, which is a contradiction. Thus p is the first conjugate point of m along γ_X, and (1) holds.

Case II. $X \neq Y$. In this case $\gamma_X \neq \gamma_Y$ and (2) holds.

THEOREM 3.3. *The mapping c is continuous over* S.

Proof. Let $X \in S$, and $\{X_k\}$ be a sequence of points of S such that limit $X_k = X$. Set $c(X_k) = a_k$. We may assume, by taking a subsequence if necessary, that limit $\{a_k\}$ exists in $R^+ \cup \{\infty\}$. Denote this limit by a. Then

$$a = c(X) \veebar a \neq c(X) .$$

We are going to prove that $a \neq c(X)$ is impossible. Hence $a = c(X)$, and c is continuous at $X \in S$. Since X is arbitrary, this proves that c is continuous over S.

Let us first assume that $c(X) > a$. Then,

(1) $$\gamma_X(a) \text{ is not conjugate to } m \text{ along } \gamma_X ,$$

and

(2) $$\gamma_X(a) \notin K_m \quad \text{along} \quad \gamma_X .$$

From (1) it follows that exp is non-singular at aX. Hence, there exists a neighborhood U of aX in $T(M)_m$ on which exp is a diffeomorphism. As $\{a_k X_k\}$ converges to aX, we may assume, by omitting a finite number of $a_k X_k$ if necessary, that all of $a_k X_k$ are in U. Since exp is a diffeomorphism from U onto exp U, it follows that $\gamma_k(a_k)$ cannot be conjugate to m along γ_k, where

$$\gamma_k = \{(t, \exp tX_k) \mid t \in [0, a_k]\} .$$

Noting that $\gamma_k(a_k) \in K_m$ along γ_k, it follows from theorem (2) that there exists another minimizing geodesic σ_k from m to $\gamma_k(a_k)$, namely

$$\sigma_k = \{(t, \exp tY_k) \mid t \in [0, a_k], Y_k \in S\} .$$

We have to note that, for every k,

$$Y_k \neq X_k \quad , \quad \gamma_k(a_k) = \sigma_k(a_k) \quad , \quad a_k Y_k \notin U .$$

By taking a subsequence if necessary, we may assume that $\{Y_k\}$ converges to some point $Y \in S$. Then $aY \notin U$ and the geodesic

$$\gamma_Y = \{(t, \exp tY) \mid t \in [0, a]\}$$

is a minimal segment from m to $\gamma_X(a) = \gamma_Y(a)$. Hence, both γ_X and γ_Y are minimal segments from m to $\gamma_X(a) = \gamma_Y(a)$. From (2) and Theorem (1) we can see that this is impossible. Hence $c(X) > a$ is false.

Let us now assume that $c(X) < a$, and let b be a positive number such that $a > c(X) + b$. Set $c(X) + b = a'$. As $\{a_k\}$ converges to a, we may assume, by omitting a finite number of a_k if necessary that $a_k > a'$, for all k.

Since $\gamma_X(a') \notin K_m$ along γ_X, it follows from Theorem (1) that there exists a unique minimal segment from m to $\gamma_X(a')$. This means that there exists a point $X' \in S$ such that $X' \neq X$ and

$$\gamma_{X'} = \{(t, \exp tX') \mid 0 \leq t \leq c(X) + b', b' < b\}$$

is a minimal segment from m to $\gamma_X(a')$. We have to note that

$$\gamma_X(a') = \gamma_{X'}(c(X) + b') .$$

We set $2r = b - b'$. It is clear that

$$\text{limit } \gamma_k(a') = \gamma_X(a') .$$

Hence we may assume, by omitting a finite number of X_k if necessary, that there exists a neighborhood U of X such that, for every k,

$$X_k \in U \quad , \quad d(\gamma_X(a'), \gamma_k(a')) < r .$$

Let α be a minimal segment from $\gamma_X(a')$ to $\gamma_k(a')$. For each fixed k, consider the curve τ from m to $\gamma_k(a')$ defined by

$$\tau = \begin{cases} \exp tX' & 0 \leq t \leq c(X) + b' \\ \alpha & \end{cases}$$

Hence,

$$L(\tau) < c(X) + b' + r = c(X) + b - r < L(\gamma_k) ,$$

where $L(\gamma_k)$ is the length of γ_k from m to $\gamma_k(a')$. This means that the geodesic segment of γ_k from m to $\gamma_k(a')$ is not minimizing. This contradicts the inequality $a_k > a'$. Hence the assumption $c(X) < a$ is false. This completes the proof.

From the continuity of c it follows immediately that the function f is continuous over S_0. Also, from the continuity of c and the exponential map it follows that g is continuous over S_0. We also have that

COROLLARY 1. *The map*

$$h_m : S_0 \to R^+$$

defined as $h_m(X) = d(m, g(X))$ *is continuous over* S_0.

4. THE CUT LOCUS OF A COMPACT MANIFOLD

For $X \in S$, let

$$E_X = \{ Y \mid Y = tX , t \in [0, c(X)) \} .$$

The set $E = \cup E_X$, for all $X \in S$, is an open cell in $T(M)_m$ called the interior set in $T(M)_m$. It is clear that $E \cap \widetilde{K}_m = \varnothing$.

THEOREM 4.1. $\exp \mid E : E \to \exp E$ *is a diffeomorphism.*

Proof. It is clear that exp is one-to-one onto exp E. For every $X \in E$, $\exp X \notin K_m$. From assertion (B) it follows that exp X is not conjugate to m for every $X \in E$. Hence dexp is non-singular at every $X \in E$. This completes the proof.

The set E is such that:

(1) exp is a diffeomorphism of E onto an open neighborhood of m in M, namely exp E.

(2) E is star shaped in the sense that if $Y \in E$ then $tY \in E$, $t \in [0, 1]$.

Hence E is a normal neighborhood of the origin zero in $T(M)_m$, and exp E is a normal neighborhood of m in M, see [2]. In fact E and exp E are the largest normal neighborhoods of zero in $T(M)_m$ and of m in M respectively.

Since $E \cap \tilde{K}_m = \varnothing$, it follows that $(\exp E) \cap K_m = \varnothing$. Let $B = E \cup \tilde{K}_m$, then

THEOREM 4.2. $\exp B = M$.

Proof. For any $p \in M$, let $d(m, p) = b$ and

$$\gamma_X = \{(t, \exp tX) \mid X \in S, 0 \leq t \leq b\}$$

be a minimal segment from m to p. Then, $b \leq c(X)$ and therefore $bX \in B$. Hence,

$$p = \exp bX \in \exp B.$$

Hence, $\exp B = M$.

From this it follows directly that $M = (\exp E) \cup K_m$. Hence K_m is a closed subset of M and $\tilde{K}_m$ is a closed subset of $T(M)_m$.

THEOREM 4.3. *The manifold* M *is compact if, and only if,* $S_0 = S$.

Proof. Suppose M is compact, and let d be the diameter of M. If $b > d$, then

$$\gamma_X = \{(t, \exp tX) \mid t \in [0, b], X \in S\}$$

is not a minimal segment from m to $\gamma_X(b)$. Hence, $c(X) \leq d$. Thus $X \in S_0$ and $S = S_0$.

Conversely, if $S_0 = S$. Then from the continuity of h_m, it follows that B is closed and bounded in $T(M)_m$, and hence compact. But then $M = \exp B$ is compact.

As an immediate consequence of this theorem it follows that.

COROLLARY 1. *If every geodesic ray from m has a conjugate point of m, then* M *is compact.*

COROLLARY 2. M *is compact if, and only if, the function f is a homeomorphism of* S_0 *onto* $\tilde{K}_m$.

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