Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Udai Pratap Singh, Prakash Chandra Yadav

On hyper Darboux lines of a Finsler hypersurface from the standpoint of the non-linear connections

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.5, p. 730–738.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_5_730_0>

Geometria differenziale. — *On hyper Darboux lines of a Finsler hypersurface from the standpoint of the non-linear connections.* Nota di UDAI PRATAP SINGH e PRAKASH CHANDRA YADAV, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Il Singh aveva già introdotto [3] (**) un'estensione delle linee di Darboux (hyper D-lines) alla ipersuperficie di uno spazio di Riemann. Tale nozione viene ora estesa alle ipersuperficie di uno spazio di Finsler servendosi della connessione non–lineare indotta su di esse. Vengono pure esaminate alcune proprietà di queste nuove curve in relazione ad un campo di vettori uscenti dai punti della ipersuperficie (union hyper D-lines).

1. INTRODUCTION

Consider an n–dimensional Finsler space F_n associated with a local coordinate system x^i $(i = 1, \cdots, n)$. Then the theory of non-linear connections in this space have been investigated by Kawaguchi [1], Rund [2] and Singh [4].

We give below some fundamental formulae which will be used in the subsequent sections of this paper. Let X^i be a vector field, $g_{ij}(x, X)$ be the components of the metric tensor of F_n and $Y_i = g_{ij}(x, X) X^j$. Suppose we are given two functions $\overset{1}{\Gamma}{}^i_k(x, X)$ and $\overset{2}{\Gamma}_{ik}(x, Y)$ such that the absolute differentials

$$\overset{1}{\delta} X^i = dX^i + \overset{1}{\Gamma}{}^i_k(x, X)\, dx^k \tag{1.1}$$

and

$$\overset{2}{\delta} Y_i = dY_i - \overset{2}{\Gamma}_{ik}(x, Y)\, dx^k \tag{1.2}$$

are respectively the components of contravariant and covariant vectors. The functions $\overset{1}{\Gamma}{}^i_k(x, X)$, $\overset{2}{\Gamma}_{ik}(x, Y)$ are supposed to be positively homogeneous of first degree in X and Y and these are used in defining the connection parameters

$$\overset{1}{\Gamma}{}^i_{jk}(x, X) = \frac{\partial \overset{1}{\Gamma}{}^i_k(x, X)}{\partial X^j} \quad , \quad \overset{2}{\Gamma}{}^i_{jk}(x, Y) = \frac{\partial \overset{2}{\Gamma}_{jk}(x, Y)}{\partial Y_i} . \tag{1.3}$$

We mention the following two conditions:

(A) If X^i undergoes parallel displacement (i.e. $\overset{1}{\delta} X^i = 0$) then so does Y_i (i.e. $\overset{2}{\delta} Y_i = 0$). This condition is characterised by (Rund [2], p. 238)

$$\overset{2}{\Gamma}_{ik}(x, Y) = \frac{\partial g_{ij}(x, X)}{\partial x^k} X^j - g_{ij}(x, X)\, \overset{1}{\Gamma}{}^j_k(x, X) . \tag{1.4}$$

(*) Nella seduta del 12 maggio 1973.

(**) Numbers in square brackets refer to the references at the end of the paper.

(B) The connection defined by $\overset{1}{\Gamma}{}^i_k(x, X)$ is metric i.e. the length of the vector field X^i remains unchanged under parallel displacement. In other words

$$\delta(g_{ij}(x, X) X^i X^j) = 0 \qquad \text{for} \quad \overset{1}{\delta} X^i = 0$$

which yields

$$\delta g_{ij}(x, X) X^i X^j = 0 . \tag{1.5}$$

This condition is characterised by ([2], p. 239)

$$Y_i \overset{1}{\Gamma}{}^i_k(x, X) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} X^i X^j. \tag{1.6}$$

Let $F_{n-1} : x^i = x^i(u^\alpha)$, $i = 1, \cdots, n$; $\alpha = 1, \cdots, n-1$ be a hypersurface of F_n and $C : u^\alpha = u^\alpha(s)$ or $x^i = x^i(s)$ be an arbitrary curve of F_{n-1} or F_n. The components X^i, X^α of a vector field of F_n and F_{n-1} are related by

$$X^i = B^i_\alpha X^\alpha \qquad \text{where} \quad B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} . \tag{1.7}$$

The metric tensors $g_{ij}(x, X)$ and $g_{\alpha\beta}(u, X)$ of F_n and F_{n-1} are such that

$$g_{\alpha\beta}(u, X) = g_{ij}(x, X) B^i_\alpha B^j_\beta . \tag{1.8}$$

There exists a vector N^i normal to F_{n-1} which satisfies the relations

$$(a) \quad g_{ij}(x, X) B^i_\alpha N^j = 0 \quad , \quad (b) \quad g_{ij}(x, X) N^i N^j = 1 . \tag{1.9}$$

Defining the induced differential of X^α as $\overset{1}{\delta} X^\alpha = B^\alpha_i \overset{1}{\delta} X^i$ where $B^\alpha_i = g^{\alpha\beta} g_{ij}(x, X) B^j_\beta$ and putting

$$\overset{1}{\delta} X^\alpha = dX^\alpha + \overset{1}{\Gamma}{}^\alpha_\gamma(u, X) du^\gamma$$

it has been shown in [4] that

$$\overset{1}{\Gamma}{}^\alpha_\gamma(u, X) = B^\alpha_i (B^i_{\beta\gamma} X^\beta + \overset{1}{\Gamma}{}^i_k B^k_\gamma) \tag{1.10}$$

where

$$B^i_{\beta\gamma} = \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma} .$$

The function $\overset{1}{\Gamma}{}^\alpha_\gamma(u, X)$ which is positively homogeneous of degree one in X may be used in defining its induced connection parameter

$$\overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u, X) = \frac{\partial \overset{1}{\Gamma}{}^\alpha_\gamma(u, X)}{\partial u^\beta} . \tag{1.11}$$

It has been shown [4] that the hypersurface F_{n-1} is metric if and only if F_n is metric.

The differential $\overset{1}{\delta} g_{ij}(x, X)$ has been given in [2] and we may write as

$$\frac{\overset{1}{\delta} g_{ij}}{\delta s} = B_{ij} = 2 C_{ijh}(x, X) \frac{\overset{1}{\delta} X^h}{\delta s} + Y_h \frac{\partial \overset{1}{\Gamma}{}^h_{ik}}{\partial X^j} \frac{dx^k}{ds}. \tag{1.12}$$

The components

$$q^i = \frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h), \quad p^\alpha = \frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma)$$

of the first curvature vectors of F_n and F_{n-1} respectively are related by [4]

$$Y_\alpha = g_{\alpha\beta}(u, X) X^\beta \quad , \quad Y_i = B^\alpha_i Y_\alpha, \tag{1.13}$$

where

$$\overline{\Omega}_{\beta\gamma}(u, X) = N_j (B^j_{\beta\gamma} + \overset{1}{\Gamma}{}^j_{hk}(x, X) B^h_\beta B^k_\gamma), \tag{1.14 a}$$

$$\hat{\Omega}_{\beta\gamma}(u, X) = N^h g_{jl} (\overset{1}{\Gamma}{}^j_{hk}(x, X) - \overset{1}{\Gamma}{}^j_{kh}(x, X)) B^l_\beta B^k_\gamma \tag{1.14 b}$$

and

$$\underset{N}{K} = \Delta_{\beta\gamma}(u, X) X^\beta X^\gamma \overset{\text{def}}{=} (\overline{\Omega}_{\beta\gamma}(u, X) + \hat{\Omega}_{\beta\gamma}(u, X)) X^\beta X^\gamma \tag{1.14 c}$$

is the normal curvature of the hypersurface.

After putting $X^i = \eta^i_{(0)} = dx^i/ds$ and taking $\eta^i_{(1)}$ as unit vector along q^i we have the following two Frenet's formulae [5]

$$q^i = \underset{(1)}{K} \underset{(1)}{\eta^i} \qquad \text{where} \quad \underset{(1)}{K} = (g_{ij}(x, X) q^i q^j)^{1/2} \tag{1.15}$$

and

$$\frac{\overset{1}{\delta} \underset{(1)}{\eta^i}}{\delta s} = -(\underset{(1)}{K} + \Omega_{(001)}) \underset{(0)}{\eta^i} - \frac{1}{2} B_{hk} \underset{(1)}{\eta^h} \underset{(1)}{\eta^k} \underset{(1)}{\eta^i} + \underset{(2)}{K} \underset{(2)}{\eta^i} \tag{1.16}$$

where it has been defined

$$\overset{1}{\Omega}{}^r_{kl}(x, X) = (\overset{1}{\Gamma}{}^r_{kl}(x, X) - \overset{1}{\Gamma}{}^r_{lk}(x, X)),$$

$$\overset{1}{\Omega}_{kml}(x, X) = g_{mr}(x, X) \overset{1}{\Omega}{}^r_{kl}(x, X) \quad , \quad \Omega_{(00)h} = \Omega_{ijh}(x, X) X^i X^j,$$

$$\Omega_{(001)} = \Omega_{hkl} \underset{(0)}{\eta^h} \underset{(0)}{\eta^k} \underset{(1)}{\eta^l} \quad \text{and} \quad \underset{(0)}{\eta^h} \;,\; \underset{(1)}{\eta^k} \;,\; \underset{(2)}{\eta^i}$$

are unit tangent, principal normal and first binormal vectors and $\underset{(1)}{K}, \underset{(2)}{K}$ are the first and second curvatures of the curve with respect to F_n.

2. HYPER D-LINE

Consider a congruence of curves given by the vector field λ^i such that through each point of F_{n-1} there passes exactly one curve of the congruence. At the points of the hypersurface we have

$$\lambda^i = t^\alpha B^i_\alpha + DN^i. \tag{2.1}$$

The curve C is said to be a hyper Darboux line of the hypersurface if the surface spanned by the vectors $\underset{(0)}{\eta}^i$ and $\underset{(1)}{R}\,\underset{(1)}{\eta}^i + \underset{(2)}{R}\,\frac{d\underset{(1)}{R}}{ds}\,\underset{(2)}{\eta}^i$ contains the vector $\lambda^i\left(\underset{(1)}{R} = \frac{1}{\underset{(1)}{K}} \text{ and } \underset{(2)}{R} = \frac{1}{\underset{(2)}{K}}\right.$ being the radii of curvatures of the first and second orders respectively$\left.\right)$. The first two Frenet's formulae yield

$$\frac{\delta \overset{1}{q}{}^i}{\delta s} = \frac{d\underset{(1)}{K}}{ds}\,\underset{(1)}{\eta}^i - \underset{(1)}{K}{}^2\,\underset{(0)}{\eta}^i - \underset{(1)}{K}\,\Omega_{(001)}\,\underset{(0)}{\eta}^i + \underset{(1)}{K}\,\underset{(2)}{K}\,\underset{(2)}{\eta}^i - \frac{1}{2}\,\underset{(1)}{K}\,B_{hk}\,\underset{(1)}{\eta}^h\,\underset{(1)}{\eta}^k\,\underset{(1)}{\eta}^i. \tag{2.2}$$

From this equation and the fact

$$g_{ij}(x\,,\,X)\,\underset{(0)}{\eta}^i\,\underset{(1)}{\eta}^J = 0$$

we obtain

$$g_{ij}(x\,,\,X)\,\frac{\delta \overset{1}{q}{}^i}{\delta s}\left(\underset{(1)}{R}\,\underset{(1)}{\eta}^j + \underset{(2)}{R}\,\underset{(2)}{\eta}^j\,\frac{d\underset{(1)}{R}}{ds}\right) = -\frac{1}{2}\,B_{hk}\,\underset{(1)}{\eta}^h\,\underset{(1)}{\eta}^k. \tag{2.3}$$

For the hyper D-line we have

$$\lambda^i = A\left(\underset{(1)}{R}\,\underset{(1)}{\eta}^i + \underset{(2)}{R}\,\frac{d\underset{(1)}{R}}{ds}\,\underset{(2)}{\eta}^i\right) + B\underset{(0)}{\eta}^i. \tag{2.4}$$

Multiplying (2.4) by $g_{ij}(x\,,\,X)\,\underset{(0)}{\eta}^j\,,\,g_{ij}(x\,,\,X)\,\underset{(1)}{\eta}^j\,,\,g_{ij}(x\,,\,X)\,\frac{\delta \overset{1}{q}{}^j}{\delta s}$ respectively, using (2.3) and writing $\underset{(h)}{\lambda} = g_{ij}(x\,,\,X)\,\lambda^i\,\underset{(h)}{\eta}^j$ $(h = 0\,,\,1\,,\,2)$ we obtain

$$B = \underset{(0)}{\lambda} \quad , \quad A = \underset{(1)}{\lambda}\,\underset{(1)}{K} \tag{2.5}$$

and

$$g_{ij}(x\,,\,X)\,\lambda^i\,\frac{\delta \overset{1}{q}{}^i}{\delta s} = -\frac{1}{2}\,\underset{(1)}{\lambda}\,\underset{(1)}{K}\,B_{hk}\,\underset{(1)}{\eta}^h\,\underset{(1)}{\eta}^k + \underset{(0)}{\lambda}\,g_{ij}(x\,,\,X)\,\frac{\delta \overset{1}{q}{}^j}{\delta s}\,\underset{(0)}{\eta}^i. \tag{2.6}$$

In view of (2.2) and (2.6) we get

$$\underset{(1)}{\lambda}\,\frac{d\underset{(1)}{K}}{ds} + \underset{(2)}{\lambda}\,\underset{(1)}{K}\,\underset{(2)}{K} = 0. \tag{2.7}$$

This yields

THEOREM (2.1). *A hyper D–line of the hypersurface satisfies the equation* (2.7).

THEOREM (2.2). *If the congruence* λ^i *is along the first curvature vector of the curve then the necessary condition that it be a hyper D–line is that it is a curve of the constant first curvature.*

Proof. The proof immediately follows from (2.7), the definition of $\underset{(1)}{\lambda}, \underset{(2)}{\lambda}$ and the fact that $\underset{(1)}{\eta^i}$ and $\underset{(2)}{\eta^i}$ are orthogonal.

3. HYPER D–LINE WITH SECOND FUNDAMENTAL TENSOR.

It has been shown in [4] that

$$\overset{1}{\Gamma}{}^{\alpha}_{\beta\gamma}(u, X) = 2 N_l \overset{1}{\Gamma}{}^l_k(x, X) B^k_\gamma M^\alpha_\beta + B^\alpha_l (B^l_{\beta\gamma} + \overset{1}{\Gamma}{}^l_{hk} B^h_\beta B^k_\gamma) + \tag{3.1}$$
$$+ 2 N_l B^l_{\varepsilon\gamma} X^\varepsilon M^\alpha_\beta$$

where

$$C_{ijk}(x, X) = \frac{1}{2} \frac{\partial g_{ij}(x, X)}{\partial X^k} \quad , \quad M_{\alpha\beta}(u, X) = C_{ijk} B^i_\alpha B^j_\beta N^k,$$

$$M^\alpha_\beta(u, X) = g^{\alpha\gamma} M_{\beta\gamma}(u, X) .$$

On multiplying (3.1) by $g_{ij}(x, X) B^j_\delta B^i_\alpha$ and using [2] $B^i_\alpha B^\alpha_l = (\delta^i_l - N^i N_l)$ we get

$$g_{ij} I^i_{\beta\gamma} B^j_\delta = - 2 N_l \overset{1}{\Gamma}{}^l_k B^k_\gamma M_{\beta\delta} - 2 N_l B^l_{\varepsilon\gamma} X^\varepsilon M_{\beta\delta} \tag{3.2}$$

where

$$I^i_{\beta\gamma} = B^i_{\beta\gamma} - B^i_\alpha \overset{1}{\Gamma}{}^\alpha_{\beta\gamma} + \overset{1}{\Gamma}{}^i_{hk} B^h_\beta B^k_\gamma . \tag{3.3}$$

Now by definition

$$\overset{1}{\delta} B^i_\alpha = \frac{\partial B^i_\alpha}{\partial u^\beta} du^\beta + B^j_\alpha \overset{1}{\Gamma}{}^i_{jh} dx^h - B^i_\gamma \overset{1}{\Gamma}{}^\gamma_{\alpha\beta} du^\beta$$

whence we may write

$$\frac{\overset{1}{\delta} B^i_\alpha}{\delta s} = I^i_{\alpha\beta} \frac{du^\beta}{ds} = s^\beta_\alpha B^i_\beta + D_\alpha N^i \tag{3.4}$$

where s^β_α, D_α are determined as follows.

Multiplying (3.4) by $g_{ij}(x, X) B^j_\delta$ and $g_{ij}(x, X) N^j$ respectively and using equations (3.2), (3.3) we get

$$s_{\alpha\delta} = - 2 \left(N_l \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M_{\alpha\delta} - 2 \left(N_l B^l_{\varepsilon\gamma} X^\varepsilon \frac{du^\gamma}{ds}\right) M_\alpha \delta \tag{3.5}$$

and

$$D_\alpha = \overline{\Omega}_{\alpha\beta} \frac{du^\beta}{ds} \tag{3.6}$$

where $\overline{\Omega}_{\alpha\beta}$ are the component of the second fundamental tensor given by (1.14 a).

From (3.4), (3.5) and (3.6) we have

$$\frac{\overset{1}{\delta} B^i_\alpha}{\delta s} = -2\left(N_l \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M^\beta_\alpha B^i_\beta - -2\left(N_l B^l_{\varepsilon\gamma} X^\varepsilon \frac{du^\gamma}{ds}\right) M^\beta_\alpha B^i_\beta + \overline{\Omega}_{\alpha\beta} \frac{du^\beta}{ds} N^i. \tag{3.7}$$

Again we decompose $\overset{1}{\delta} N^i/\delta s$ as follows

$$\frac{\overset{1}{\delta} N^i}{\delta s} = A^\alpha B^i_\alpha + EN^i. \tag{3.8}$$

This equation yields

$$g_{ij} \frac{\overset{1}{\delta} N^i}{\delta s} B^j_\beta = A_\beta . \tag{3.9}$$

Operating (1.9 *a*) by $\overset{1}{\delta}/\delta s$ and using (1.12), (3.7), (3.9) we find

$$A^\alpha = - B_{rj} g^{\alpha\beta} N^r B^j_\beta - g^{\alpha\beta} \overline{\Omega}_{\beta\gamma} \frac{du^\gamma}{ds} . \tag{3.10}$$

Also we may easily deduce from (3.8) and (1.9 *b*) that

$$E = g_{ij} \frac{\overset{1}{\delta} N^i}{\delta s} N^j = -\frac{1}{2} B_{rj} N^i N^j. \tag{3.11}$$

Substituting in (3.8), the value of A^α from (3.10) and E from (3.11) we get

$$\frac{\overset{1}{\delta} N^i}{\delta s} = -\left(B_{rj} g^{\alpha\beta} N^r B^j_\beta + g^{\alpha\beta} \overline{\Omega}_{\beta\gamma} \frac{du^\gamma}{ds}\right) B^i_\alpha - \frac{1}{2} B_{rj} N^r N^j N^i. \tag{3.12}$$

From (3.7), (3.12) and the relation obtained after operating (1.13) by $\overset{1}{\delta}/\delta s$ we have

$$\frac{\overset{1}{\delta} q^i}{\delta s} = \left(\frac{\overset{1}{\delta} p^\alpha}{\delta s} - \underset{N}{K} B_{rj} g^{\alpha\beta} N^r B^j_\beta - \underset{N}{K} g^{\alpha\beta} \overline{\Omega}_{\beta\gamma} \frac{du^\gamma}{ds}\right) B^i_\alpha + \left(\overline{\Omega}_{\alpha\beta} p^\alpha \frac{du^\beta}{ds} - \frac{1}{2} \underset{N}{K} B_{rj} N^r N^j + \frac{d\underset{N}{K}}{ds}\right) N^i - 2\left(N_l \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M^\beta_\alpha B^i_\beta p^\alpha - 2\left(N_l B^l_{\varepsilon\gamma} X^\varepsilon \frac{du^\gamma}{ds}\right) M^\beta_\alpha B^i_\beta p^\alpha. \tag{3.13}$$

Putting the value of $\overset{1}{\delta} q^i/\delta s$ from (3.13) and λ^i from (2.1) in the equation (2.6) we obtain

$$(3.14)\qquad g_{\alpha\beta}(u\,,\,X)\,\frac{\overset{1}{\delta} p^\alpha}{\delta s}\,t^\beta - \underset{N}{K} B_{rj}\, N^r\, B^j_\beta\, t^\beta - \underset{N}{K}\overline{\Omega}_{\alpha\beta}\, t^\alpha \frac{du^\beta}{ds} +$$

$$+ D\left(\overline{\Omega}_{\alpha\beta}\, p^\alpha \frac{du^\beta}{ds} - \frac{1}{2}\underset{N}{K} B_{rj}\, N^r N^j + \frac{d\underset{N}{K}}{ds}\right) - 2\left(N_l\, \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M_{\alpha\beta}\, p^\alpha\, t^\beta -$$

$$- 2\left(N_l\, B^l_{\varepsilon\gamma}\, X^\varepsilon \frac{du^\gamma}{ds}\right) p^\alpha M_{\alpha\beta}\, t^\beta + \frac{1}{2}\,\lambda\, \underset{(1)(1)}{K B_{hk}}\, \underset{(1)}{\eta^h}\, \underset{(1)}{\eta^k} -$$

$$- \underset{(0)}{\lambda}\left[g_{\alpha\beta}(u\,,\,X)\,\frac{\overset{1}{\delta} p^\alpha}{\delta s}\frac{du^\beta}{ds} - \underset{N}{K} B_{rj}\, N^r\, B^j_\beta \frac{du^\beta}{ds} - \underset{N}{K}\overline{\Omega}_{\alpha\beta}\frac{du^\alpha}{ds}\frac{du^\beta}{ds} -\right.$$

$$\left. - 2\left(N_l\, \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M_{\alpha\beta}\, p^\alpha \frac{du^\beta}{ds} - 2\left(N_l\, B^l_{\varepsilon\gamma}\, X^\varepsilon \frac{du^\gamma}{ds}\right) M_{\alpha\beta}\, p^\alpha \frac{du^\beta}{ds}\right] = 0$$

where we have used the fact that $\underset{(0)}{\eta^i}$, is orthogonal to N^i.

Further, the first two Frenet's formulae with respect to F_{n-1} read as

$$(3.15)\qquad p^\alpha = \underset{(1)}{k}\, \underset{(1)}{\xi^\alpha}$$

and

$$(3.16)\qquad \frac{\overset{1}{\delta}\underset{(1)}{\xi^\alpha}}{\delta s} = -\left(\underset{(1)}{k} + \overset{*}{\Omega}_{(001)}\right)\underset{(0)}{\xi^\alpha} - \frac{1}{2}\, B_{\beta\gamma}\, \underset{(1)}{\xi^\beta}\, \underset{(1)}{\xi^\gamma}\, \underset{(1)}{\xi^\alpha} + \underset{(2)}{k}\, \underset{(2)}{\xi^\alpha}$$

where we have taken the definitions

$$\overset{1}{\overset{*}{\Omega}}{}^\gamma_{\delta\beta}(u\,,\,X) = \left(\overset{1}{\Gamma}{}^\gamma_{\delta\beta}(u\,,\,X) - \overset{1}{\Gamma}{}^\gamma_{\beta\delta}(u\,,\,X)\right),\ \overset{1}{\overset{*}{\Omega}}_{\delta\alpha\beta}(u\,,\,X) = g_{\alpha\gamma}(u\,,\,X)\,\overset{1}{\overset{*}{\Omega}}{}^\gamma_{\delta\beta}(u\,,\,X)\,,$$

$$\overset{*}{\Omega}_{(00)\gamma} = \overset{*}{\Omega}_{\alpha\beta\gamma}(u\,,\,X)\,\underset{(0)}{\xi^\alpha}\,\underset{(0)}{\xi^\beta}\quad,\quad \overset{*}{\Omega}_{(001)} = \overset{*}{\Omega}_{\alpha\beta\gamma}\,\underset{(0)}{\xi^\alpha}\,\underset{(0)}{\xi^\beta}\,\underset{(1)}{\xi^\gamma},$$

$$B_{\alpha\beta} \equiv \frac{\overset{1}{\delta} g_{\alpha\beta}}{\delta s}\qquad \text{and}\quad \underset{(0)}{\xi^\alpha} = \frac{du^\alpha}{ds}\,,\ \underset{(1)}{\xi^\alpha}\,,\ \underset{(2)}{\xi^\alpha}$$

are unit tangent, unit principal normal, unit first binormal vectors and $\underset{(1)}{k}\,,\,\underset{(2)}{k}$ are the first and second curvatures of the curve with respect to the hypersurface.

These two Frenet's formulae with respect to hypersurface yield

$$(3.17)\qquad \frac{\overset{1}{\delta} p^\alpha}{\delta s} = \frac{d\underset{(1)}{k}}{ds}\,\underset{(1)}{\xi^\alpha} - \underset{(1)}{k^2}\,\underset{(0)}{\xi^\alpha} - \underset{(1)}{k}\,\overset{*}{\Omega}_{(001)}\,\underset{(0)}{\xi^\alpha} + \underset{(1)}{k}\,\underset{(2)}{k}\,\underset{(2)}{\xi^\alpha} - \frac{\underset{(1)}{k}}{2}\, B_{\beta\gamma}\,\underset{(1)}{\xi^\beta}\,\underset{(1)}{\xi^\gamma}\,\underset{(1)}{\xi^\alpha}.$$

Substituting the value of $\overset{1}{\delta} p^\alpha / \delta s$ in (3.14) and using the relations

$$\underset{(0)}{\lambda} = g_{ij}(x\,,\,X)\,\lambda^i \underset{(0)}{\eta^j} = g_{\alpha\beta}\,(u\,,\,X)\,t^\alpha \frac{du^\beta}{ds} = \underset{(0)}{t}\,,$$

$$\underset{(1)}{\lambda}\,\underset{(1)}{K} = g_{ij}(x\,,\,X)\,\lambda^i q^j = g_{\alpha\beta}(u\,,\,X)\,t^\alpha p^\beta + D\underset{N}{K}$$

we get

$$\begin{aligned}(3.18)\qquad & \frac{d\underset{(1)}{k}}{ds}\underset{(1)}{t} + \underset{(1)}{k}\,\underset{(2)}{k}\,\underset{(2)}{t} - \frac{\underset{(1)}{k}}{2} B_{\alpha\beta}\,\underset{(1)}{\xi^\alpha}\,\underset{(1)}{\xi^\beta}\,\underset{(1)}{t} - \underset{N}{K}\,B_{rj}\,N^r B^j_\beta t^\beta - \\ & - \underset{N}{K}\,\overline{\Omega}_{\alpha\beta}\,t^\alpha \frac{du^\beta}{ds} + D\left(\overline{\Omega}_{\alpha\beta}\,p^\alpha \frac{du^\beta}{ds} - \frac{1}{2}\underset{N}{K}\,B_{rj}\,N^r N^j + \frac{d\underset{N}{K}}{ds}\right) \\ & - 2\left(N_l \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M_{\alpha\beta}\,p^\alpha t^\beta - 2\left(N_l B^l_{\varepsilon\gamma} X^\varepsilon \frac{du^\gamma}{ds}\right) M_{\alpha\beta}\,p^\alpha t^\beta + \\ & + \frac{1}{2}\left(\underset{(1)}{k}\,\underset{(1)}{t} + D\underset{N}{K}\right) B_{hk}\,\underset{(1)}{\eta^h}\,\underset{(1)}{\eta^k} + \underset{(0)}{t}\left[\underset{N}{K}\,B_{rj}\,N^r B^j_\beta \frac{du^\beta}{ds} + \right. \\ & + \underset{N}{K}\,\overline{\Omega}_{\alpha\beta} \frac{du^\alpha}{ds}\frac{du^\beta}{ds} + 2\left(N_l \overset{1}{\Gamma}{}^l_k \frac{dx^k}{ds}\right) M_{\alpha\beta}\,p^\alpha \frac{du^\beta}{ds} + \\ & \left. + 2\left(N_l B^l_{\varepsilon\gamma} X^\varepsilon \frac{du^\gamma}{ds}\right) M_{\alpha\beta}\,p^\alpha \frac{du^\beta}{ds}\right] = 0\end{aligned}$$

where we have used

$$\underset{(1)}{t} = g_{\alpha\beta}(u\,,\,X)\,t^\alpha \underset{(1)}{\xi^\beta} \quad ; \quad \underset{(2)}{t} = g_{\alpha\beta}(u\,,\,X)\,t^\alpha \underset{(2)}{\xi^\beta}.$$

The equation (3.18) represents the hyper D–line of the hypersurface.

4. Hyper D–line and union curves

A curve of the hypersurface is called union curve relative to the congruence λ^i if surface spanned by the vectors dx^i/ds and q^i contains λ^i. For this curve we have

$$(4.1)\qquad \lambda^i = A\underset{(0)}{\eta^i} + Bq^i \qquad \text{where} \quad \underset{(0)}{\eta^i} = X^i = \frac{dx^i}{ds}\,.$$

From (1.11), (2.1), (3.15) and (4.1), we obtain

$$t^\alpha = A\frac{du^\alpha}{ds} + Bp^\alpha \quad , \quad D = B\underset{N}{K} \quad , \quad \underset{(0)}{t} = A \quad , \quad \underset{(1)}{t} = B\underset{(1)}{k} \quad , \quad \underset{(2)}{t} = 0\,.$$

Substituting these values in (3.18) we get, after some simplification,

$$
(4.2)\qquad \underset{(1)}{k}\frac{d\underset{(1)}{k}}{ds} - \frac{1}{2} B_{\alpha\beta}\, p^{\alpha} p^{\beta} - \underset{N}{K}\, B_{rj}\, N^{r}\, B^{j}_{\beta}\, p^{\beta} - \frac{1}{2}\underset{N}{K}^{2} B_{rj}\, N^{r} N^{j} +
$$

$$
\underset{N}{K}\frac{d\underset{N}{K}}{ds} - 2\left(N_{l}\,\overset{1}{\Gamma}{}^{l}_{k}\,\frac{dx^{k}}{ds}\right) M_{\alpha\beta}\, p^{\alpha} p^{\beta} -
$$

$$
-2\left(N_{l}\, B^{l}_{\varepsilon\gamma}\, X^{\varepsilon}\,\frac{du^{\gamma}}{ds}\right) M_{\alpha\beta}\, p^{\alpha} p^{3} + \frac{1}{2} B_{hk}\, q^{h} q^{k} = 0\,.
$$

Now operating the relation (1.8) by $\overset{1}{\delta}/\delta s$ and using (1.12), (3.7) we find

$$
(4.3)\qquad \frac{\overset{1}{\delta} g_{\alpha\beta}}{\delta s} \equiv B_{\alpha\beta} = B_{ij}\, B^{i}_{\alpha}\, B^{j}_{\beta} - 4\left(N_{l}\,\overset{1}{\Gamma}{}^{l}_{k}\,\frac{dx^{k}}{ds}\right) M_{\alpha\beta} - 4\left(N_{l}\, B^{l}_{\varepsilon\gamma}\, X^{\varepsilon}\,\frac{du^{\gamma}}{ds}\right) M_{\alpha\beta}\,.
$$

Using (1.13), (4.2) and (4.3) we have

$$
(4.4)\qquad \underset{(1)}{k}\frac{d\underset{(1)}{k}}{ds} + \underset{N}{K}\frac{d\underset{N}{K}}{ds} = 0
$$

which after integration gives

$$
(4.5)\qquad \underset{(1)}{k}^{2} + \underset{N}{K}^{2} = C^{2}
$$

where C^2 is the constant of integration. Hence we can establish the following theorems.

THEOREM (4.1). *The sum of the squares of the geodesic and of the normal curvatures in the direction of a union hyper D–line is same at all the points of the curve.*

THEOREM (4.2). *The sum of the squares of the geodesic and of the normal curvatures in the direction of the union hyper D–line is same relative to every congruence* λ^i.

REFERENCES

[1] KAWAGUCHI A., *On the theory of non-linear connections*, «Tensor N.S.», *6*, 165–199 (1956).

[2] RUND H., *The differential geometry of Finsler spaces*, Springer–Verlag (1959).

[3] SINGH U. P., *On hyper Darboux lines in Riemannian sub-space*, «Tensor N.S.», *20*, 1–4 (1969).

[4] SINGH U. P., *On the induced theory of Finsler hypersurfaces from the standpoint of non-linear connection*, «Atti della Accad. Naz. Lincei Rendiconti» (to appear).

[5] SINGH U. P., *Frenet's formulae associated with non-linear connections* (under publication).