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Induced and intrinsic derivatives on the subspace of special Kawaguchi space

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Geometria differenziale. — *Induced and intrinsic derivatives on the subspace of special Kawaguchi space.* Nota di UDAI PRATAP SINGH e SHRI KRISHNA DEO DUBEY, presentata [(*)] dal Socio E. BOMPIANI.

RIASSUNTO. — Nella teoria degli spazi speciali di Kawaguchi esistono due tipi di connessione (indotta e intrinseca) su una varietà immersa (come nella geometria di Finsler). La loro differenza è stata determinata da Yoshida [2] [(1)].

In questa Nota si definiscono e studiano due tipi di vettori normali di curvatura. Si discutono inoltre i due tipi di parallelismo di un campo vettoriale.

1. INTRODUCTION

We shall consider an n–dimensional metric space Kn in which the arc length of a curve $x^i = x^i(t)$ [(2)] is given by following integral:

$$S = \int [A_i(x^i, \overset{\prime}{x}{}^i)\, \overset{\prime\prime}{x}{}^i + B(x^i, \overset{\prime}{x}{}^i)]^{1/p}\, dt \,, \tag{1.1}$$

where

$$\overset{\prime}{x}{}^i = \frac{dx^i}{dt} \quad , \quad \overset{\prime\prime}{x}{}^i = \frac{d^2 x^i}{dt^2}$$

and A_i, B are differentiable functions of x^i and $\overset{\prime}{x}{}^i$. If $p = 3$, n is even then the space Kn is called an n–dimensional special Kawaguchi space of order 2. The theory of such a space was established by A. Kawaguchi [1] and was studied by several authors.

In order that the arc length in the space should remain unaltered under any transformation of the parameter t, we must have the so-called Zermelo's conditions,

$$A_i \overset{\prime}{x}{}^i = 0 \quad , \quad A_{i(j)} \overset{\prime}{x}{}^j = (p-2) A_i \quad , \quad B_{(i)} \overset{\prime}{x}{}^i = pB \,, \tag{1.2}$$

where

$$A_{i(j)} = \frac{\partial A_i}{\partial \overset{\prime}{x}{}^j} \quad , \quad B_{(i)} = \frac{\partial B}{\partial \overset{\prime}{x}{}^i} \,.$$

And, since (1.1) is scalar, it follows that A_i is a vector.

Let us consider an m–dimensional subspace Km of Kn represented by the equations $x^i = x^i(u^\alpha)$ and the matrix of the projection factors $p^i_\alpha = \dfrac{\partial x^i}{\partial u^\alpha}$ has rank m. If we denote a_α and b the quantities in Km corresponding to A_i

(*) Nella seduta del 12 maggio 1973.

(1) Numbers in the brackets refer to the references at the end of the paper.

(2) Throughout this paper, Latin indices run from 1 to n, Greek ones $\alpha, \beta, \gamma, \delta, \varepsilon$ from 1 to m and μ, ν from $m+1$ to n.

and B in Kn respectively, it follows that the same equations concerning a_α and b as (1.2) hold.

Putting

$$(1.3) \qquad G_{ij} = 2\, A_{i(j)} - A_{j(i)} \quad , \quad G_{\alpha\beta} = 2\, a_{\alpha(\beta)} - a_{\beta(\alpha)} \, ,$$

it has been shown ([2]),

$$(1.4) \qquad G_{ij}\, p^i_\alpha\, p^j_\beta = G_{\alpha\beta} \, .$$

Suppose $G^{\alpha\beta}$ is the tensor reciprocal to $G_{\alpha\beta}$ (i.e. $G^{\beta\alpha}\, G_{\beta\gamma} = \delta^\alpha_\gamma$) Yoshida [2] defines

$$(1.5) \qquad p^\alpha_i = G^{\alpha\beta}\, G_{ij}\, p^j_\beta$$

on the assumption that n, m are both even and det (G_{ij}) $\{$det $(G_{\alpha\beta})\}$ does not vanish identically. It is easy to show that

$$p^\alpha_i\, p^i_\beta = \delta^\alpha_\beta \, .$$

The connections Γ^i of Kn and Γ^α of Km are given as

$$(1.6) \qquad 2\, \Gamma^i = (2\, A_{lm}\, \acute{x}^m - B_{(l)})\, G^{li} \quad , \quad 2\, \Gamma^\alpha = (2\, a_{\beta\gamma}\, \acute{u}^\gamma - b_{(\beta)})\, G^{\beta\alpha}.$$

The covariant differential of a contravariant vector field $v^i(x^i, \acute{x}^i)$ homogeneous of degree zero with respect to the $\acute{x}^i$ is defined by Kawaguchi [1]

$$(1.7) \qquad \delta v^i = dv^i + \Gamma^i_{jk}\, v^j\, dx^k$$

where

$$\Gamma^i_{jk} = \frac{\partial^2\, \Gamma^i}{\partial \acute{x}^j\, \partial \acute{x}^k} = \Gamma^i_{kj} \, .$$

If v^α is a vector field in Km such that $v^i = p^i_\alpha\, v^\alpha$, then the induced covariant differential $\check{\delta} v^\alpha$ is defined as (Yoshida [2])

$$(1.8) \qquad \check{\delta} v^\alpha = p^\alpha_i\, \delta v^i \, .$$

Putting

$$\check{\delta} v^\alpha = dv^\alpha + \check{\Gamma}^\alpha_{\beta\gamma}\, v^\beta\, du^\gamma \, ,$$

it has been shown in [2]

$$(1.9) \qquad \check{\Gamma}^\alpha_{\beta\gamma} = p^\alpha_i\, (p^i_{\beta\gamma} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\gamma) \, .$$

Yoshida ([3]) defines for the projection parameters p^i_α,

$$(1.10) \qquad \overset{\circ}{H}{}^i_{\beta\alpha} \overset{\text{def}}{=} \overset{\circ}{D}_\beta\, p^i_\alpha \overset{\text{def}}{=} p^i_{\alpha\beta} + \Gamma^i_{jk}\, p^j_\beta\, p^k_\alpha - \check{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma \, .$$

Using (1.9) and normal vector $\underset{\mu}{n}{}^i$ of Kn, (1.10) can be written as

$$(1.11) \qquad \overset{\circ}{H}{}^i_{\beta\alpha} = H^\mu_{\beta\alpha}\, \underset{\mu}{n}{}^i$$

from which it follows

$$H^{\mu}_{\beta\alpha} = G^{\nu\mu} G_{ij} \underset{\nu}{n}^{i} \overset{\circ}{H}{}^{j}_{\beta\alpha} = G^{\mu\nu} \underset{\nu}{n}_{i} \overset{\circ}{H}{}^{i}_{\beta\alpha} , \tag{1.12}$$

where $G^{\nu\mu}$ has the same meaning as in [2].

2. Difference between induced and intrinsic connections

The connection coefficients of the imbedding space Kn are denoted by Γ^{i}_{jk} and the induced connection coefficients of subspace Km have been given by (1.9). With the help of (1.5) and (1.9) we can write,

$$G_{\alpha\delta} \check{\Gamma}^{\alpha}_{\beta\gamma} = G_{ij} p^{j}_{\delta} (p^{i}_{\beta\gamma} + \Gamma^{i}_{jk} p^{j}_{\beta} p^{k}_{\gamma}) . \tag{2.1}$$

The most significant motivation for the definition (1.9) is in the fact that it leads directly to the fundamentally important induced normal curvature vector $\overset{\circ}{H}{}^{i}_{\beta\alpha}$, given by (1.10), which derives its name from the circumstance that it is normal to Km, i.e.

$$G_{ij} p^{j}_{\delta} \overset{\circ}{H}{}^{i}_{\beta\alpha} = 0 \tag{2.2}$$

as is immediately evident by multiplication of (1.10) by $G_{ij} p^{j}_{\delta}$ and subsequent application of (1.4) and (2.1).

The intrinsic connection coefficients $\Gamma^{\alpha}_{\beta\gamma} \overset{\text{def}}{=} \dfrac{\partial^2 \Gamma^{\alpha}}{\partial \acute{u}^{\beta} \partial \acute{u}^{\gamma}}$ have been defined as ([2])

$$\Gamma^{\alpha}_{\beta\gamma} = \check{\Gamma}^{\alpha}_{\beta\gamma} + \Gamma^{*i}_{(\beta)} p^{\alpha}_{i\,(\gamma)} + \Gamma^{*i}_{(\gamma)} p^{\alpha}_{i\,(\beta)} + \Gamma^{*i} p^{\alpha}_{i\,(\beta)\,(\gamma)} , \tag{2.3}$$

where

$$\Gamma^{*i} \overset{\text{def}}{=} \Gamma^{i} + \frac{1}{2} p^{i}_{\alpha\beta} \acute{u}^{\alpha} \acute{u}^{\beta} .$$

In analogy with (1.10) we construct the intrinsic curvature vector

$$\overset{\circ}{\overset{*}{H}}{}^{i}_{\beta\alpha} \overset{\text{def}}{=} p^{i}_{\alpha\beta} + \Gamma^{i}_{jk} p^{j}_{\beta} p^{k}_{\alpha} - \Gamma^{\gamma}_{\alpha\beta} p^{i}_{\gamma} . \tag{2.4}$$

Let us define the form,

$$\Lambda_{\beta\varepsilon\gamma} \overset{\text{def}}{=} \Gamma_{\beta\varepsilon\gamma} - \check{\Gamma}_{\beta\varepsilon\gamma} = G_{\alpha\varepsilon} (\Gamma^{\alpha}_{\beta\gamma} - \check{\Gamma}^{\alpha}_{\beta\gamma}) . \tag{2.5}$$

This equation and (2.3) yield

$$\Lambda^{\alpha}_{\beta\gamma} = p^{\alpha}_{i\,(\gamma)} \Gamma^{*i}_{(\beta)} + p^{\alpha}_{i\,(\beta)} \Gamma^{*i}_{(\gamma)} + \Gamma^{*i} p^{\alpha}_{i\,(\beta)\,(\gamma)} . \tag{2.6}$$

In [2] Γ^{α} has been written in the form,

$$2\,\Gamma^{\alpha} = [2\,\Gamma^{i} + p^{i}_{\beta\gamma} \acute{u}^{\beta} \acute{u}^{\gamma}]\, p^{\alpha}_{i}$$

which gives

$$p^i_\alpha \Gamma^\alpha = \Gamma^i + \frac{1}{2} p^i_{\beta\gamma} u'^\beta u'^\gamma = \Gamma^{*i} . \tag{2.7}$$

If we use (2.7), then the equation (2.6) can be expressed as

$$\begin{aligned} \Lambda^\alpha_{\beta\gamma} &= \Gamma^\alpha_{(\beta)(\gamma)} - \Gamma^{*i}_{(\beta)(\gamma)} p^\alpha_i \\ &= \Gamma^\alpha_{(\beta)(\gamma)} - \left(p^i_\alpha \Gamma^\alpha\right)_{(\beta)(\gamma)} p^\alpha_i , \end{aligned} \tag{2.8}$$

since $p^i_\alpha \Gamma^\alpha = \Gamma^{*i}$, the above equation becomes

$$\Lambda^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} - \Gamma^{*i}_{\beta\gamma} p^\alpha_i . \tag{2.9}$$

If $A_{i(j)(k)} = 0$ and m, n are both even, then it has been proved in [2] that the intrinsic and induced connection parameters coincide, i.e.

$$\Lambda^\alpha_{\beta\gamma} = 0 . \tag{2.10}$$

Now substituting (1.10) from (2.4) and using (2.5), we get

$$\overset{\circ}{\overset{*}{H}}{}^i_{\beta\alpha} = \overset{\circ}{H}{}^i_{\beta\alpha} - p^i_\delta \Lambda^\delta_{\beta\alpha} . \tag{2.11}$$

Hence, we get

THEOREM (2.1). *The intrinsic curvature vector differs from induced normal curvature vector merely by a tangential component.*

Further the equation (2.11) proves:

COROLLARY (2.1). *The vector* $\overset{\circ}{\overset{*}{H}}{}^i_{\beta\alpha}$ *is not, in general, normal to the subspace.*

If in analogy to equation (1.12) we define

$$\overset{*}{H}{}^\mu_{\beta\alpha} = \overset{\circ}{\overset{*}{H}}{}^i_{\beta\alpha} G_{ij} \underset{\nu}{n}{}^j G^{\mu\nu} \tag{2.12}$$

then in view of the equations (2.11) and (2.12) we obtain,

$$\overset{*}{H}{}^\mu_{\beta\alpha} = H^\mu_{\beta\alpha} . \tag{2.13}$$

Therefore,

THEOREM (2.2). *The same second fundamental tensor can be used in both the induced and intrinsic theories.*

3. DERIVATIVES OF A VECTOR FIELD

For the curve $C : x^i = x^i(t)$ [or $u^\alpha = u^\alpha(t)$] of the subspace Km, the unit tangent vectors $x'^i = \frac{dx^i}{dt}$ and $u'^\alpha = \frac{du^\alpha}{dt}$ are related by

$$x'^i = p^i_\alpha u'^\alpha .$$

Let

$$X^i(t) = p^i_\alpha U^\alpha(t) \tag{3.1}$$

be a vector field tangent to Km. It is supposed that X^i is a differentiable function of $(x^j, \dot{x}^j)$. We shall now obtain a relation between the covariant derivative $\delta X^i/\delta t$ (with respect to the connection of the imbedding space Kn) and the intrinsic covariant derivative $\delta U^\alpha/\delta t$ (with respect to the intrinsic connection of Km). The latter is defined by

$$\frac{\delta U^\varepsilon}{\delta t} = \frac{dU^\varepsilon}{dt} + \Gamma^\varepsilon_{\alpha\beta} U^\alpha \dot{u}^\beta . \tag{3.2}$$

The equation (3.1), when differentiated with respect to t along the curve C, yields,

$$\frac{dX^i}{dt} = p^i_{\alpha\beta} U^\alpha \dot{u}^\beta + p^i_\varepsilon \frac{dU^\varepsilon}{dt} . \tag{3.3}$$

Also, due to (1.7) and (3.1), we have

$$\frac{\delta X^i}{\delta t} = \frac{dX^i}{dt} + \Gamma^i_{jk} U^\alpha p^j_\alpha \dot{u}^\beta p^k_\beta . \tag{3.4}$$

The elimination of $p^i_{\alpha\beta}$ from (3.3) and (2.4) gives

$$\frac{dX^i}{dt} = \left[\overset{\circ}{\overset{*}{H}}{}^i_{\beta\alpha} - \Gamma^i_{jk} p^j_\beta p^k_\alpha + \Gamma^\gamma_{\alpha\beta} p^i_\gamma\right] U^\alpha \dot{u}^\beta + p^i_\varepsilon \frac{dU^\varepsilon}{dt} .$$

After simplifying this equation with the help of (3.2) and (3.4), we get

$$\frac{\delta X^i}{\delta t} = \overset{\circ}{\overset{*}{H}}{}^i_{\beta\alpha} U^\alpha \dot{u}^\beta + p^i_\varepsilon \left(\frac{\delta U^\varepsilon}{\delta t}\right) \tag{3.5}$$

which in view of (2.11) and (1.11) takes the form,

$$\frac{\delta X^i}{\delta t} = H^\mu_{\beta\alpha} \underset{\mu}{n}{}^i U^\alpha \dot{u}^\beta + p^i_\varepsilon \left(\frac{\delta U^\varepsilon}{\delta t} - \Lambda^\varepsilon_{\beta\alpha} U^\alpha \dot{u}^\beta\right) . \tag{3.6}$$

This is the required result.

The following theorems are immediate from equation (3.6):

THEOREM (3.1). *If the vector field* X^i *tangential to* Km *is parallel along a curve (of* Km*) with respect to* Kn*, it is not necessary that the vector field* U^α *is parallel along the curve with respect to intrinsic connection of* Km*. Under these circumstances, we have*

$$\frac{\delta U^\varepsilon}{\delta t} = \Lambda^\varepsilon_{\beta\alpha} U^\alpha \dot{u}^\beta . \tag{3.7}$$

THEOREM (3.2). *If the induced and intrinsic connection parameters coincide then the parallelism (along a curve of* Km*) of the vector field* X^i *in* Kn *implies the parallelism along the curve of the vector field* U^α *in* Km *and vice versa.*

4. INDUCED DERIVATIVES

In analogy to article 3, we shall derive an expression involving the covariant derivative $\delta X^i/\delta t$ of a vector field X^i and induced covariant derivative $\check{\delta} U^\alpha/\check{\delta} t$. The latter is given by

$$\frac{\check{\delta} U^\varepsilon}{\check{\delta} t} = \frac{dU^\varepsilon}{dt} + \check{\Gamma}^\varepsilon_{\alpha\beta}\, U^\alpha\, \acute{u}^\beta. \tag{4.1}$$

On eliminating $p^i_{\alpha\beta}$ from (1.10) and (3.3), we get

$$\frac{dX^i}{dt} = \left[\overset{\circ}{H}{}^i_{\beta\alpha} - \Gamma^i_{jk}\, p^j_\beta\, p^k_\alpha + \check{\Gamma}^\gamma_{\alpha\beta}\, p^i_\gamma\right] U^\alpha\, \acute{u}^\beta + p^i_\varepsilon \frac{dU^\varepsilon}{dt}\,.$$

Taking the help of equations (3.4) and (4.1) this equation becomes,

$$\frac{\delta X^i}{\delta t} = \overset{\circ}{H}{}^i_{\beta\alpha}\, U^\alpha\, \acute{u}^\beta + p^i_\varepsilon \frac{\check{\delta} U^\varepsilon}{\check{\delta} t}\,. \tag{4.2}$$

Using (1.11), we get

$$\frac{\delta X^i}{\delta t} = \underset{\mu}{n}{}^i H^\mu_{\beta\alpha}\, U^\alpha\, \acute{u}^\beta + p^i_\varepsilon \frac{\check{\delta} U^\varepsilon}{\check{\delta} t}\,. \tag{4.3}$$

This yields the following:

THEOREM (4.1). *The necessary and sufficient condition that a vector field of the subspace be parallel along a curve in* K*n* *is that it is parallel along the curve with respect to induced connection of* K*m*.

Now we suppose that $U^\alpha = \acute{u}^\alpha$. On the assumption that Γ^i are homogeneous of degree two with respect to the $\acute{x}$, it has been shown (Yoshida [2]),

$$\Gamma^\varepsilon_{\beta\alpha}\, \acute{u}^\beta\, \acute{u}^\alpha = \check{\Gamma}^\varepsilon_{\beta\alpha}\, \acute{u}^\beta\, \acute{u}^\alpha\,. \tag{4.4}$$

Hence, by (2.5) we get

$$\Lambda^\varepsilon_{\beta\alpha}\, \acute{u}^\beta\, \acute{u}^\alpha = 0\,. \tag{4.5}$$

Thus, Theorem (3.1) takes the form:

THEOREM (4.2). *The necessary and sufficient condition that a curve of the subspace be auto parallel in* K*n*, *is that, it is auto parallel in* K*m* *with respect to the intrinsic connection parameter.*

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