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Mauro Beltrametti

Some remarks about the contact of hypersurfaces along multiple subvarieties

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Geometria. — *Some remarks about the contact of hypersurfaces along multiple subvarieties.* Nota di MAURO BELTRAMETTI, presentata (*) dal Socio E. G. TOGLIATTI.

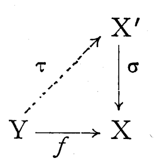
RIASSUNTO. — Estendendo ricerche di D. Gallarati [2] e di molti altri Autori sul contatto tra due ipersuperficie d'uno spazio proiettivo S_r lungo una varietà ad $r-2$ dimensioni, si considera il caso in cui il contatto avvenga lungo le varie falde d'una varietà multipla per le due ipersuperficie.

1. First, we mention some definitions and known results to be used in the sequel (see also [3], par. 8).

Let us denote by (X, O_X), (Y, O_Y) two algebraic varieties defined on a field k, by $\mathfrak{I}$ a sheaf of ideals of O_X, by $f: Y \to X$ a morphism and by $f^*: O_X \to O_Y$ the associated morphism between the structure sheaves.

DEFINITION (1). We say that f makes the ideal $\mathfrak{I}$ invertible (or divisorial) if $f^{-1}\mathfrak{I}$ is an invertible sheaf on Y, where $f^{-1}\mathfrak{I}$ is the ideal generated by the image of $f^*\mathfrak{I}$ in O_Y.

DEFINITION (2). Denoting by $\sigma: X' \to X$ a morphism which makes $\mathfrak{I}$ invertible, the pair (X', σ) is said to be a *blow up of* X *along* $\mathfrak{I}$ if for every morphism $f: Y \to X$, which makes $\mathfrak{I}$ invertible, there exists one and only one morphism $\tau: Y \to X'$ such that the diagram:



commutes.

PROPOSITION (1). *Let* X *be a variety and* $\mathfrak{I}$ *a sheaf of ideals of* O_X, $\mathfrak{I} \neq 0$. *Then there exists a blow up* (X', σ) *of* X *along* $\mathfrak{I}$ *which is unique up to isomorphisms.*

DEFINITION (3). Let Z be a closed subset of X defined by the ideal $\mathfrak{I}_Z$ of O_X. The blow up of X along $\mathfrak{I}_Z$ is called *blow up of* X *along* Z. In particular if Z is a point P we get blow up of X at P.

PROPOSITION (2). *If* X *and* Z *are non singular* X' *is non singular, moreover* $\dim X = \dim X'$.

Last, we observe that if (X', σ) is the blow up of X along $\mathfrak{I}$ and if U is a nonempty open set of X, then $(\sigma^{-1}(U), \sigma/\sigma^{-1}(U))$ is the blow up of $(U, \mathfrak{I}/U)$.

(*) Nella seduta del 12 maggio 1973.

Let X be a non singular, irreducible, complete algebraic variety of dimension n, defined over an algebraically closed field k of characteristic zero, let D be an hypersurface of X, V a non singular, irreducible, subvariety of X of codimension $\lambda + 1$. $\mathfrak{I}_V$ the sheaf of ideals of O_X defining V. Further, we denote by P a point of V, by U an affine neighborhood of P in X, by $f = 0$, with $f \in \Gamma(U, O_X)$, a local equation of D in U and write $\mathfrak{M}_V = \Gamma(U, \mathfrak{I}_V)$. We set the following:

DEFINITION (4). D passes through V with multiplicity s if:

$$f \in \mathfrak{M}_V^s \quad , \quad f \notin \mathfrak{M}_V^{s+1}.$$

Let us now assume that D passes through V with multiplicity s and that there exists an effective divisor Δ of V such that the generic point of every irreducible component δ of Δ has multiplicity s_1 (1) with respect to D ($s_1 > s$). Thus if $\mathfrak{I}_\delta$ is the sheaf of ideals defining δ and $\mathfrak{M}_\delta = \Gamma(U, \mathfrak{I}_\delta)$ we have:

$$(1) \qquad f \in \mathfrak{M}_V^s \quad , \quad f \notin \mathfrak{M}_V^{s+1} \quad ; \quad f \in \mathfrak{M}_\delta^{s_1} \quad , \quad f \notin \mathfrak{M}_\delta^{s_1+1}.$$

By the hypothesis made, it is possible to choose the affine neighborhood U such that V and δ become complete intersection on U; hence we get:

$$\mathfrak{M}_V = (g_1, \cdots, g_{\lambda+1}) \quad , \quad \mathfrak{M}_\delta = (g_1, \cdots, g_{\lambda+2}) \quad , \quad g_i \in \Gamma(U, O_X).$$

Let $(\sigma^{-1}(U), \sigma/\sigma^{-1}(U))$ be the blow up of U along $V \cap U$, induced by the blow up (X', σ) of X along V. In the variety $U \times \mathbf{P}^\lambda$ we consider the closed subset U′ defined by the equations:

$$g_i X_j - g_j X_i = 0 \qquad\qquad i, j = 1, \cdots, \lambda + 1.$$

where $X_1, \cdots, X_{\lambda+1}$ are homogeneous coordinates in $\mathbf{P}^\lambda$. Outside of V the projection $\sigma : U' \to U$ is an *isomorphism* and $U' = \sigma^{-1}(U)$.

Now we prove that $\mathfrak{I}_V$ *becomes divisorial* on U′. To achieve this purpose take over $U \times \mathbf{P}^\lambda$ the open set $U \times U_i$, with $U_i = \operatorname{Spec}\left(k\left[\frac{X_1}{X_i}, \cdots, \frac{X_{\lambda+1}}{X_i}\right]\right)$. U′ is defined on $U \times U_i$ by the equations: $g_j = g_i \frac{X_j}{X_i}$, $j = 1, \cdots, \lambda+1$ (i fixed); hence on $U' \cap (U \times U_i)$ the ideal $\sigma^{-1}\mathfrak{I}_V$ is generated by g_i. Thus g_i defines on $U' \cap (U \times U_i)$ the *exceptional divisor* $E = \sigma^{-1}(V)$. One gets:

$$U \times U_i = \operatorname{Spec}\left(\Gamma(U, O_X)\left[\frac{X_1}{X_i}, \cdots, \frac{X_{\lambda+1}}{X_i}\right]\right)$$

so that:

$$U' \cap (U \times U_i) = \operatorname{Spec}\left(\Gamma(U, O_X)\left[\frac{X_1}{X_i}, \cdots, \frac{X_{\lambda+1}}{X_i}\right] \Big/ \left(\cdots, g_j - g_i \frac{X_j}{X_i}, \cdots\right)\right).$$

Putting $U_{g_i} = U - \{g_i = 0\}$ it follows:

$$U_i' = U' \cap (U_{g_i} \times U_i) = \operatorname{Spec}\left(\Gamma(U, O_X)\left[\frac{g_1}{g_i}, \cdots, \frac{g_{\lambda+1}}{g_i}\right]\right).$$

(1) Shortly denoted, in the sequel, by s_1-point.

2. Let $f = 0$ be a local equation of D in U; owing to the relations (1) there exist $\alpha_s, \alpha_{s+1}, \cdots, \alpha_{s_1}$ with $\alpha_j \in \mathfrak{M}_\nu^j$ $(j = s, \cdots, s_1)$, $\alpha_s \notin \mathfrak{M}_\nu^{s+1}$ and α_s homogeneous such that:

$$(2) \qquad f - \left(g_{\lambda+2}^{s_1-s}\alpha_s + g_{\lambda+2}^{s_1-s-1}\alpha_{s+1} + \cdots + g_{\lambda+2}\,\alpha_{s_1-1} + \alpha_{s_1}\right) \in \mathfrak{M}_\nu^{s_1+1}.$$

On the open set U_{g_i} write $g_k = g_i \dfrac{g_k}{g_i}$ $(k = 1, \cdots, \lambda + 2)$: since $\alpha_j \in \mathfrak{M}_\nu^j$ we get, for every j, $\alpha_j = g_i^j \beta_j$, where

$$\beta_j \in \left(\frac{g_1}{g_i}, \cdots, \frac{g_{\lambda+1}}{g_i}\right)^j \Gamma(U, O_X).$$

Hence in U_{g_i} a relation of the form:

$$(3) \qquad f = g_i^s \left[g_{\lambda+2}^{s_1-s}\beta_s + g_i \cdot \eta\right] = 0$$

holds, where g_i does not divide β_s and where η is element of $\mathfrak{M}_\nu^{s_1}$.

In the open set U'_i we have the following: $g_i = 0$ is an equation of the *exceptional divisor* E, $g_{\lambda+2}^{s_1-s} \cdot \beta_s + g_i \cdot \eta$ stands for the *proper transform* $D^{(1)}$ of D, $g_{\lambda+2} = 0$ is an equation of $\delta^* = \sigma^{-1}(\delta) \in \mathrm{Div}(E)$. Moreover, if V_1 is the divisor of E locally represented by the equation $\beta_s = 0$ (2), we get on E:

$$E \cdot D^{(1)} = V_1 + (s_1 - s)\,\delta^*,$$

(taking into account that the open sets U'_i are a covering of U').

Therefore, we have proved the following

PROPOSITION (3). $O_{X'}(D^{(1)}) \otimes_{O_{X'}} O_E = O_E(V_1) \otimes_{O_{X'}} O_E((s_1 - s)\,\delta^*)$.

Remark: $(g_1, \cdots, g_{\lambda+1})$ is a $\Gamma(U, O_X)$ sequence and, if $P \notin \Delta$, it may be completed in $O_{X,P}$ to a $O_{X,P}$ sequence $(g'_1, \cdots, g'_{\lambda+1}, \cdots, g'_n)$ formed by a regular system of parameters, where g'_i is the image of g_i in $O_{X,P}$, $i = 1, \cdots, \lambda+1$. Hence g'_α, $\alpha = 1, \cdots, n$ may be viewed as indeterminates [1].

3. Let D_1, D_2 be two hypersurfaces of X passing with multiplicity s through the subvariety V of X. Hereinafter we shall assume that an open set U_V of V exists in which D_1 and D_2 have s distinct branches: we mean that for every point P of U_V it is possible to find a surface intersecting D_1 and D_2 in two curves, each having at P s distinct tangent lines (i.e. P is an *ordinary* s–point for the two section curves). Moreover, we assume the existence of two divisors δ_1 and δ_2 of V, loci (3) of s_1 and s_2–points for D_1 and D_2 respectively $(s_1 > s, s_2 > s)$.

(2) Clearly V_1 does not depend on the choice of α_s.

(3) (i.e. the generic point of every irreducible component of δ_i is s_i–point, $i = 1, 2$).

If in the affine open set U the divisor D_i has equation $f_i = 0$ $(i = 1, 2)$, we write, with the notation of the previous section:

$$f_1 = g_i^s \left[g_{\lambda+2}^{s_1 - s} \beta_s^{(1)} + g_i \cdot \eta_1\right] = 0 \tag{4}$$

$$f_2 = g_i^s \left[g_{\lambda+2}^{s_2 - s} \beta_s^{(2)} + g_i \cdot \eta_2\right] = 0 .$$

If the divisors V_1, V_2 of E, locally represented on U_i' by $\beta_s^{(1)} = 0$, $\beta_s^{(2)} = 0$, coincide then $\alpha_s^{(1)} = g_i^s \beta_s^{(1)}$ and $\alpha_s^{(2)} = g_i^s \beta_s^{(2)}$ coincide up to invertible factors. *This means that in every point* $P \in V$ *not belonging to* $\operatorname{Supp} \delta_1 \cup \operatorname{Supp} \delta_2$, D_1 *and* D_2 *have the same tangent cone.* In fact, on account of remark of Section 2 we get, for every such point P:

$$\widehat{O}_{X,P} \overset{\xi}{\simeq} k[[g_1', \cdots, g_n']] .$$

On the other hand the tangent cone to D_1 at P is the affine scheme:

$$\operatorname{Spec}(gr\, O_{D_1,P}) = \operatorname{Spec}(gr\, O_{X,P}/(f_1)\, O_{X,P}) \simeq$$

$$\simeq \operatorname{Spec}(gr\, \widehat{O}_{X,P}/(f_1)\, \widehat{O}_{X,P}) \simeq \operatorname{Spec}\left(gr\, k[[g_1', \cdots, g_n']]/(\xi(f_1))\right) \simeq$$

$$\simeq \operatorname{Spec}\left(gr\, k[[g_1', \cdots, g_n']]/(\overline{\xi(f_1)})\right),$$

where $\overline{\xi(f_1)}$ is the lower-degree term of $\xi(f_1)$; since $\alpha_s^{(1)}, \alpha_s^{(2)}$ are homogeneous: $\overline{\xi(f_1)} = \rho \overline{\xi(f_2)}$, $\rho \in k$, is equivalent to $V_1 = V_2$.

4. We denote $D_1^{(1)}$ and $D_2^{(1)}$ the proper transforms of the divisors D_1 and D_2 while δ_1^* and δ_2^* stand for the inverse images of δ_1 and δ_2 $(\delta_i \in \operatorname{Div}(V)$, $i = 1, 2)$.

Formulas (4) yield:

$$O_{X'}(D_1^{(1)}) \otimes_{O_{X'}} O_E = O_E\,(V_1 + (s_1 - s)\, \delta_1^*)$$

and

$$O_{X'}(D_2^{(1)}) \otimes_{O_{X'}} O_E = O_E\,(V_2 + (s_2 - s)\, \delta_2^*) .$$

Whenever D_1 and D_2 have the same tangent cone at the points of V not belonging to $\operatorname{Supp} \delta_1 \cup \operatorname{Supp} \delta_2$, i.e. in the hypothesis $V_1 = V_2$, one has:

$$O_{X'}(D_1^{(1)} - D_2^{(1)}) \otimes_{O_{X'}} O_E = O_E\left[(s_1 - s)\, \delta_1^* - (s_2 - s)\, \delta_2^*\right] .$$

Suppose V_1 be a prime, non singular divisor (this is motivated by the generality of $\alpha_s^{(1)}$) (4).

(4) The argument which follows can be extended, with minor changes, to the case where V_1 is reducible with every component non singular, by considering every irreducible component.

Tensoring by $O_{V_1} = O_{V_2}$ on O_E and still denoting by δ_i^* the divisors intersected on V_1 by δ_i^*, $i = 1, 2$:

(5) $$O_{X'}(D_1^{(1)} - D_2^{(1)}) \otimes_{O_{X'}} O_{V_1} = O_{V_1}\left[(s_1 - s)\,\delta_1^* - (s_2 - s)\,\delta_2^*\right].$$

Let now $\bar{D}_1$ and $\bar{D}_2$ be two divisors of X such that $\bar{D}_1 \equiv D_1$, $\bar{D}_2 \equiv D_2$ and assume that $\bar{D}_1$, $\bar{D}_2$ do not contain V ([5], Cap. II, par. 3). Putting $\bar{D}_i^* = \sigma^{-1}(\bar{D}_i)$, $i = 1, 2$, it follows:

$$\bar{D}_1^* - \bar{D}_2^* \equiv D_1^{(1)} + sE - D_2^{(1)} - sE = D_1^{(1)} - D_2^{(1)};$$

whence, by use of (5):

(6) $$O_{X'}(\bar{D}_1^* - \bar{D}_2^*) \otimes_{O_{X'}} O_{V_1} = O_{V_1}\left[(s_1 - s)\,\delta_1^* - (s_2 - s)\,\delta_2^*\right].$$

This formula is equivalent to state the linear equivalence of the two divisors intersected on V_1 by $(s_1 - s)\,\delta_1^* - (s_2 - s)\,\delta_2^*$ and by $\bar{D}_1^* - \bar{D}_2^*$ respectively.

We remark that, if $(s_1 - s)\,\delta_1^* - (s_2 - s)\,\delta_2^* \equiv 0$, in particular if $\delta_1 = \delta_2 = 0$, whe have:

$$O_E\left[(\bar{D}_1^* - \bar{D}_2^*)\cdot E\right] = O_E$$

then, since $\sigma_* O_E = O_V$:

(7) $$(\bar{D}_1 - \bar{D}_2)\cdot V \equiv 0.$$

We suppose now that the subvariety V has codimension two ($\lambda = 1$). In this case there exists on V an open set U_V such that the restriction to U_V of the morphism $\tau = \sigma/V_1 : V_1 \to V$ has finite fibres ([4], Cap. I, p. 96), all of them being formed by s distinct points. Indeed if U_V is the open set of V in which D_1 and D_2 have s distinct branches and if S is a surface which intersects V in a point P and D_1, D_2 along curves having in P an ordinary s–point, the blow up of X along V induces blow up at P of both these curves; furthermore P blows up in s distinct points belonging to V_1.

In the hypothesis that the variety X has *dimension 3* and that all the points forming the divisor δ_i $(i = 1, 2)$ have the same multiplicity $s_i = s + 1$ (this situation is a general one) eq. (6) implies that the two divisors intersected on the curve V_1 by $\delta_1^* - \delta_2^*$ and by $\bar{D}_1^* - \bar{D}_2^*$ have the same degree. Since the divisors $\bar{D}_1$ and $\bar{D}_2$ may be choosen to ensure that no point $P \notin U_V$ is contained in Supp $\bar{D}_1 \cup$ Supp $\bar{D}_2$, we have:

(8) $$\deg(\bar{D}_1^* - \bar{D}_2^*)_{/V_1} = s \deg(\bar{D}_1 - \bar{D}_2)_{/V},$$

then:

$$s \deg(\bar{D}_1 - \bar{D}_2)_{/V} = \deg \delta_1^* - \deg \delta_2^*.$$

On the other hand [5]:

$$\deg \delta_1^* = s \deg \delta_1 \quad , \quad \deg \delta_2^* = s \deg \delta_2$$

whence:

$$\deg (\bar{D}_1 - \bar{D}_2)_{/V} = \deg \delta_1 - \deg \delta_2 \,. \tag{9}$$

Let us make the following remark: if D_1 and D_2 are two surfaces, of orders d_1 and d_2, belonging to $X = \mathbf{P}^3$ and if h is the order of the curve V, we have from (9):

$$h (d_1 - d_2) = \deg \delta_1 - \deg \delta_2 \,.$$

If $\delta_1 \equiv \delta_2$ (in particular if $\delta_1 = \delta_2 = 0$) it follows $d_1 = d_2$. Thus we have proved the following:

PROPOSITION (4). *Let D_1 and D_2 be two algebraic surfaces of $\mathbf{P}^3$, each passing with multiplicity s, and with s distinct branches, through a non singular curve V of order h. If δ_i is the divisor on V locus of s_i-points ($s_i = 1 + s$) for D_i ($i = 1, 2$) and if D_1, D_2 have the same tangent cone at the points of V not belonging to* Supp $\delta_1 \cup$ Supp δ_2, *then the equality* $h (d_1 - d_2) = \deg \delta_1 - \deg \delta_2$ *holds.*

COROLLARY. *Two surfaces D_1, D_2 of $\mathbf{P}^3$ both passing with multiplicity s through a non singular curve C and not having on C points of multiplicity $> s$, can have the same tangent cone along C only if their orders are equal.*

The previous proposition is extended in an obvious way to the case $X = \mathbf{P}^n$: if two hypersurfaces D_1 and D_2, of the same order, have the same tangent cone at every point $P \notin \text{Supp } \delta_1 \cup \text{Supp } \delta_2$ it follows:

$$(s_1 - s)\, \delta_1^* \equiv (s_2 - s)\, \delta_2^* \,.$$

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(5) There exists on the curve V a divisor $\bar{\delta}_1 \equiv \delta_1$ such that Supp $\bar{\delta}_1 \subset U_V$, hence $\deg \bar{\delta}_1^* = s \deg \delta_1$. But $\deg \bar{\delta}_1^* = \deg \delta_1^*$, for $\bar{\delta}_1^* \equiv \delta_1^*$; it follows: $\deg \delta_1^* = s \deg \delta_1$. The equality $\deg \delta_2^* = s \deg \delta_2$ follows similarly.