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On the numerical solution of a Goursat problem considered by M. Picone

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Analisi matematica. — *On the numerical solution of a Goursat problem considered by M. Picone* (*). Nota di KIN VINH LEUNG, DEMETRIO MANGERON (**) e MEHMET NAMIK OĞUZTÖRELI, presentata (***) dal Socio M. PICONE.

RIASSUNTO. — In questo lavoro gli Autori, prendendo le mosse da tutt'una serie di lavori dell'Illustre Accademico Linceo Mauro Picone, pubblicati più di sessant'anni fa, ma la cui portata è, a tutt'oggi, ben difficile sottovalutare, presentano uno schema numerico di risoluzione del problema di Goursat considerato da Picone in [1] e [2].

1. Let $f(x)$ and $g(y)$ be two given functions which are monotonic and of class C′ in the intervals $0 \leq x \leq \alpha$ and $0 \leq y \leq \beta$ $(\alpha, \beta > 0)$ respectively satisfying the conditions

$$(1.1) \qquad \begin{cases} f(0) = g(0) = 0 \quad , \quad 0 \leq f'(0) g'(0) < 1 \\ 0 \leq f(x) < x \quad , \quad 0 \leq g(y) < y . \end{cases}$$

Let $X(x)$ and $Y(y)$ be functions of class C′ in $0 \leq x \leq \alpha$ and $0 \leq y \leq \beta$ respectively, such that $X(0) = Y(0)$. Further let $A(x, y)$, $B(x, y)$, $C(x, y)$ and $F(x, y)$ be functions which are sufficiently smooth in the closure $\bar{R}$ of the rectangle $R: 0 < x < \alpha, 0 < y < \beta$. Clearly, the origin is the only common point to the curves $y = f(x)$ and $x = g(y)$ in $\bar{R}$.

The Goursat problem which consists of the finding of a function $u = u(x, y)$ of class C′ in $\bar{R}$ with a continuous first total derivative $\frac{\partial^2 u}{\partial x \partial y}$ satisfying the hyperbolic equation

$$(1.2) \qquad \frac{\partial^2 u}{\partial x \partial y} = A(x, y) u + B(x, y) \frac{\partial u}{\partial x} + C(x, y) \frac{\partial u}{\partial y} + F(x, y)$$

and the conditions

$$(1.3) \qquad \begin{cases} u(x, f(x)) = X(x) & \text{for } 0 \leq x \leq \alpha , \\ u(g(y), y) = Y(y) & \text{for } 0 \leq y \leq \beta , \end{cases}$$

has been considered in the pioneering work of E. Goursat [3] and E. Picard [4], and fully investigated by M. Picone in his famous Memoires [1] and [2]. M. Picone established very elegantly the unique solution of the

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Goursat problem (I.2)–(I.3) solving an equivalent integral equation of more complex structure than of the usual Volterra integral equation with two variables. The results of M. Picone were extended to D. Mangeron's polyvibrating equations by M. N. Oğuztöreli [5] (1).

In this paper we present a numerical scheme to construct the solution of the above formulated Goursat problem.

II. Let N be a sufficiently large positive integer. Put

$$h = \frac{\alpha}{N} \quad , \quad k = \frac{\beta}{N} . \tag{II.1}$$

Let us divide R into N^2 rectangular elements of size $h \times k$ by the lines $x = rh$, $y = sk$ $(r, s = 1, 2, \cdots, N-1)$. We shall denote the grid points (rh, sk) by $P_{r,s}$ or $P^{(i)}$ with

$$i = i(r, s) = (s-1)(N-1) + r , \tag{II.2}$$

and for the sake of convenience we also write $P^{(0)} = P_{0,0} = (0, 0) = 0$. Let $u = u(x, y)$ be the solution to the Goursat problem (I.2)–(I.3) whose existence and uniqueness are established by M. Picone. We wish to compute the values of $u(x, y)$ at the nodes $P_{r,s}$ $(r, s = 0, 1, \cdots, N)$.

To simplify our presentation, we shall denote the value of a function $z = z(x, y)$ at $P_{r,s}$ by one of the following notations: $z_{r,s} = z(rh, sk) = z(P_{r,s}) = z_i = z(P^{(i)})$. Further we shall use the symbols $z_i^{(m,n)}$ or $z_{r,s}^{(m,n)}$ to denote the value of $\dfrac{\partial^{m+n} z}{\partial x^m \partial y^n}$ at the node $P^{(i)}$:

$$z_i^{(m,n)} = z_{r,s}^{(m,n)} = \left(\frac{\partial^{m+n} z}{\partial x^m \partial y^n} \right)_{r,s} , \quad z_i^{(0,0)} = z_i . \tag{II.3}$$

III. We begin with the calculations of the partial derivatives $u_0^{(m,n)}$ of u at $P^{(0)}$, the origin, with $0 < m + n < 4$ $(m, n = 0, 1, 2, 3)$. For this purpose we subdivide the first rectangular element of corners $P_{0,0}$, $P_{1,0}$, $P_{0,1}$, $P_{1,1}$ into nine rectangular elements by the lines $x = \frac{h}{3}$, $x = \frac{2h}{3}$, $y = \frac{k}{3}$, $y = \frac{2k}{3}$. Let $Q_1(h_1, k_1)$, $Q_2(h_2, k_2)$ and $Q_3(h_3, k_3)$ be the curve points on $y = f(x)$ and $Q_4(h_4, k_4)$, $Q_5(h_5, k_5)$ and $Q_6(h_6, k_6)$ be the curve points on $x = g(y)$ where

$$\text{(III.1)} \quad \begin{cases} h_1 = \dfrac{h}{3} \quad , \quad h_2 = \dfrac{2h}{3} \quad , \quad h_3 = h ; \\ \qquad h_4 = g\left(\dfrac{k}{3}\right) , \; h_5 = g\left(\dfrac{2k}{3}\right) , \; h_6 = g(k) , \\ k_1 = f\left(\dfrac{h}{3}\right) \quad , \quad k_2 = f\left(\dfrac{2h}{3}\right) \quad , \quad k_3 = f(h) ; \\ \qquad k_4 = \dfrac{k}{3} \quad , \quad k_5 = \dfrac{2k}{3} \quad , \quad k_6 = k . \end{cases}$$

(1) Various other extension of Goursat problems are to be found in [9]–[10].

Let U_r be the value of u at the points Q_r $(r = 1, \cdots, 6)$. We have

$$\text{(III.2)} \qquad U_1 = X\left(\frac{h}{3}\right), \quad U_2 = X\left(\frac{2h}{3}\right), \quad U_3 = X(h);$$
$$U_4 = Y\left(\frac{k}{3}\right), \quad U_5 = Y\left(\frac{2k}{3}\right), \quad U_6 = Y(k),$$

by virtue of Eqs (I.3). Further

$$\text{(III.3)} \qquad u^{(1,1)} = Au + Bu^{(1,0)} + Cu^{(0,1)} + F$$

by Eq (I.2). By differentiation we can easily show that

$$\text{(III.4)} \qquad u^{(2,1)} = A_1^* u + B_1^* u^{(1,0)} + C_1^* u^{(0,1)} + F_1^*,$$

and

$$\text{(III.5)} \qquad u^{(1,2)} = A_2^* u + B_2^* u^{(1,0)} + C_2^* u^{(0,1)} + F_2^*$$

where

$$\text{(III.6)} \qquad \begin{cases} A_1^* = A^{(1,0)} + AC, & A_2^* = A^{(0,1)} + AB, \\ B_1^* = B^{(1,0)} + BC + A, & B_2^* = B^{(0,1)} + BB, \\ C_1^* = C^{(1,0)} + CC, & C_2^* = C^{(0,1)} + CB + A, \\ F_1^* = F^{(1,0)} + FC, & F_2^* = F^{(0,1)} + FB. \end{cases}$$

We now expand the six nodal values U_r of u at the curve points Q_r $(r = 1, \cdots, 6)$ in Taylor's series until and inclusive the third order terms

$$\text{(III.7)} \qquad U_r = u_0^{(0)} + h_r u_0^{(1,0)} + k_r u_0^{(0,1)} + $$
$$+ \frac{1}{2!}\left[h_r^2 u_0^{(2,0)} + 2 h_r k_r u_0^{(1,1)} + k_r^2 u_0^{(0,2)}\right]$$
$$+ \frac{1}{3!}\left[h_r^3 u_0^{(3,0)} + 3 h_r^2 k_r u_0^{(2,1)} + 3 h_r k_2^2 u_0^{(1,2)} + k_r^3 u_0^{(0,3)}\right].$$

Combining (III.3)–(III.7) and putting

$$\text{(III.8)} \qquad u' = [u^{(1,0)}, u^{(0,1)}, u^{(2,0)}, u^{(0,2)}, u^{(3,0)}, u^{(0,3)}],$$

$$\text{(III.9)} \qquad z = [z_1, \cdots, z_6],$$

$$z_1 = \left[U_i - \left\{1 + h_i k_i \left(A + \frac{h_i}{2} A_1^* + \frac{k_i}{2} A_2^*\right)\right\} u - h_i k_i \left(F + \frac{h_i}{2} F_1^* + \frac{k_i}{2} F_2^*\right)\right]_0$$

and $T \equiv [\alpha_{ij}]$ $(i, j = 1, \cdots, 6)$ with

$$\text{(III.10)} \qquad \begin{cases} \alpha_{i,1} = h_i \left[1 + k_i B + \frac{k_i}{2}(h_i B_1^* + k_i B_2^*)\right]_0, & \alpha_{i,4} = \frac{k_i^2}{2}(1 + h_i C)_0, \\ \alpha_{i,2} = k_i \left[1 + h_i C + \frac{h_i}{2}(h_i C_1^* + k_i C_2^*)\right], & \alpha_{i,5} = \frac{h_i^3}{6}, \\ \alpha_{i,3} = \frac{h_i^2}{2}(1 + k_i B)_0, & \alpha_{i,6} = \frac{k_i^3}{6}, \end{cases}$$

we obtain the equation $Tu' = z$ from which follows $u' = T^{-1} z$ if T^{-1} exists. Note that $u_0^{(1,1)}$, $u_0^{(2,1)}$ and $u_0^{(1,2)}$ can be derived directly from Eqs. (III.3)–(III.6).

Next we compute the values of u at the three corner nodes $P^{(1)}$, $P^{(2)}$ and $P^{(3)}$ by the expansions

$$\text{(III.11)}\quad \begin{cases} u_1 = u_{1,0} = u_0 + h u_0^{(1,0)} + \dfrac{h^2}{2!} u_0^{(2,0)} + \dfrac{h^3}{3!} u_0^{(3,0)} , \\ u_2 = u_{0,1} = u_0 + k u_0^{(0,1)} + \dfrac{k^2}{2!} u_0^{(0,2)} + \dfrac{k^3}{3!} u_0^{(0,3)} , \\ u_3 = u_{1,1} = u_1 + u_2 - u_0 + hk u_0^{(1,1)} + \dfrac{hk}{2!} \left(h u_0^{(2,1)} + k u_0^{(1,2)} \right) . \end{cases}$$

Further, we can evaluate the partial derivatives $U_3^{(m,n)}$ and $U_6^{(m,n)}$ of u at the end curve points Q_3 and Q_6 for $0 < m+n \leq 3$ by the following Taylor expansions:

$$\text{(III.12)}\quad \begin{cases} U_r^{(1,0)} = u_0^{(1,0)} + h_r u_0^{(2,0)} + k_r u_0^{(1,1)} + \\ \qquad + \dfrac{1}{2!} \left[h_r^2 u_0^{(3,0)} + 2 h_r k_r u_0^{(2,1)} + k_r^2 u_0^{(1,2)} \right] , \\ U_r^{(0,1)} = u_0^{(0,1)} \pm h_r u_0^{(1,1)} + k_r u_0^{(0,2)} + \\ \qquad + \dfrac{1}{2!} \left[h_r^2 u_0^{(2,1)} + 2 h_r k_r u_0^{(1,2)} + k_r^2 u_0^{(0,3)} \right] , \\ U_r^{(2,0)} = u_0^{(2,0)} + h_r u_0^{(3,0)} + k_r u_0^{(2,1)} , \\ U_r^{(1,1)} = \left[Au + Bu^{(1,0)} + Cu^{(0,1)} + F \right]_r , \\ U_r^{(0,2)} = u_0^{(0,2)} + h_r u_0^{(1,2)} + k_r u_0^{(0,3)} , \\ U_r^{(2,1)} = \left[A_1^* u + B_1^* u^{(1,0)} + C_1^* u^{(0,1)} + Bu^{(2,0)} + F_1^* \right]_r , \\ U_r^{(1,2)} = \left[A_1^* u + B_1^* u^{(1,0)} + C_1^* u^{(0,1)} + Bu^{(2,0)} + F_1^* \right] , \\ U_r^{(3,0)} = u_0^{(2,0)} + h_3 u_0^{(3,0)} + k_3 u_0^{(2,1)} , \\ U_r^{(0,3)} = u_0^{(0,2)} + h_3 u_0^{(1,2)} + k_3 u_0^{(0,3)} , \end{cases}$$

for $r = 3$ and 6.

IV. Now, let consider the adjoining element after relabelling the end curve points Q_3 or Q_6 as Q_0^*. Let Q_2^* be the other end curve points in this new square and Q_1^* be a curve point between Q_0^* and Q_2^*. Let u_0^*, u_1^* and u_2^* be the values of u at Q_0^*, Q_1^* and Q_2^*, respectively. We choose Q_0^* as the new origin of the coordinates. Let (h_3, k_3), (h_4, k_4) be the coordinates of the two corner nodes $P^{*(3)}$, $P^{*(4)}$ which on the opposite side to Q_0^* of the new rectangular element. We can compute $u_3^* = (P^{*(3)})$ and $u_4^* = u\,(P^{*(4)})$ by the formula

$$u_r^* = u_0^* + h_r\, u_0^{*(1,0)} + k_r\, u_0^{*(0,1)} + \tag{IV.1}$$
$$+ \frac{1}{2!}\left[h_r^2\, u_0^{*(2,0)} + 2\, h_r\, k_r\, u_0^{*(1,1)} + k_r^2\, u_0^{*(0,2)}\right] +$$
$$+ \frac{1}{3!}\left[h_r^3\, u_0^{*(3,0)} + 3\, h_r^2\, k_r\, u_0^{*(2,1)} + 3\, h_r\, k_r^2\, u_0^{*(1,2)} + k_r^3\, u_0^{*(0,3)}\right],$$

for $r = 3$ and 4.

We now can use the formulas (III.12) to find the partial derivatives of u at the end-point Q_2^*. We can proceed this process until Q_2^* reaches the boundary of the rectangle R. By then u_0 is given and $4\,N - 1$ out of $(N + 1)^2$ nodal values of u have been computed. The remaining $(N + 1)^2 - 4\,N = (N - 1)^2$ nodal values of u can be computed by the following formula which can be easily established:

$$u_{r+1,s+1} - u_{r+1,s-1} - u_{r-1,s+1} + u_{r-1,s-1} - 4\,hkA_{r,s}\,u_{r,s} - \tag{IV.2}$$
$$- 2\,kB_{r,s}\,(u_{r+1,s} - u_{r-1,s}) - 2\,hC_{r,s}\,(u_{r,s+1} - u_{r,s-1}) = 4\,hkF_{r,s}\,.$$

Note that in these equations each of $(N - 1)^2$ internal nodes $P_{r,s}$ $(r\,,\,s = 1\,,\cdots,\,N-1)$ will be successively used as center of (IV.2). The center should be chosen such that 8 out 9 nodal values of u in (IV.2) are already computed and the 9^{th} nodal value of u should be obtained by (IV.2). In this way we complete our scheme to evaluate the solution $u\,(x\,,y)$ of the Darboux problem (I.2)–(I.3) at the $(N + 1)^2$ nodal points $P_{r,s}$ $(r\,,\,s = 0\,,\,1\,,\cdots,\,N)$. Clearly, the accuracy of the above method essentially depends on the magnitude of h and k. A computer simulation of the above analysis will be presented in a separate paper where we shall also give several three dimensional illustrative figures of the solutions of some Goursat problems of the form (I.2)–(I.3). Further, in another paper, we shall extend the above method to a Goursat problem considered in [5].

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