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On a class of nonlinear integro-differential equations. Nota II

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Analisi matematica. — *On a class of nonlinear integro-differential equations.* Nota II di KIN VINH LEUNG, DEMETRIO MANGERON (*), MEHMET NAMIK OĞUZTÖRELI (**) e ROBERTO B. STEIN, presentata (***) dal Socio M. PICONE.

RIASSUNTO. — Gli Autori continuando i loro studi anteriori [1]-[3] espongono in questo lavoro le loro investigazioni concernenti una nuova classe di equazioni integro-differenziali che ne costituiscono un modello matematico riflettente certe attività elettriche delle reti neuronali.

1. In this paper we consider a nonlinear integro-differential system of the form

$$
\text{(I.1)} \quad \begin{cases} \dfrac{dx_i}{dt} = a \left[\dfrac{1}{1 + \exp\left\{ -f_i(t) - \sum\limits_{j=1}^{N} c_{ij}\, x_j - \sum\limits_{k=1}^{K} b_{ik} \displaystyle\int_0^t g_{ik}[x_i(s)]\, e^{-p_{ik}(t-s)}\, ds \right\}} - x_i \right] \\ x_i(0) = x_i^0 \qquad\qquad (i = 1, \cdots, N) \end{cases}
$$

where N and K are certain given natural numbers, a, b_{ik}, c_{ij} and p_{ik} are given constants ($a > 0$, $c_{ii} = 0$ and $p_{ik} > 0$), $f_i(t)$ and $g_{ik}(x_i)$'s are given functions which are sufficiently smooth in their arguments, x_i^0's are given constants such that $0 < x_i^0 < 1$, and $x_i = x_i(t)$'s are the unknown functions.

The differential system (I.1) with $b_{ik} = 0$ ($i = 1, \cdots, N$; $k = 1, \cdots, K$) and $1 \leq N \leq \infty$ has been investigated in [1] and [2]. In the present Note we assume that at least one of the b_{ik}'s is not zero and N is finite.

Let us note that the integro-differential system considered in [3] can be considered as a first approximation of Eqs. (I.1) for sufficiently small b_{ik}'s.

In the study of electrical activities in certain neural networks we have $a \cong 100$ and $g_{ik}(x_i) \equiv x_i$. The cases $\{N = 1, K = 1\}$ and $\{N = 1, K = 2\}$ are particularly of great importance in the study of isolated neurons. The biological implications of our results given here will be presented somewhere else.

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II. To investigate Eqs. (I.1), as in [3], we put

$$\text{(II.1)} \qquad \begin{cases} x_{i0} = x_i \\ x_{ik} = \displaystyle\int_0^t g_{ik}[x_i(s)]\, e^{-p_{ik}(t-s)}\, ds \end{cases} \qquad (i = 1, \cdots, N\,;\ k = 1, \cdots, K)\,.$$

In this way we find

$$\text{(II.2)} \qquad \begin{cases} \dfrac{dx_{i0}}{dt} = a\left[\dfrac{1}{1 + \exp\left\{-f_i(t) - \sum\limits_{j=1}^{N} c_{ij}\, x_{j0} - \sum\limits_{k=1}^{K} b_{ik}\, x_{ik}\right\}} - x_{i0}\right] \\ \dfrac{dx_{ik}}{dt} = g_{ik}(x_i) - p_{ik}\, x_{ik} \end{cases}$$

subject to the conditions

$$\text{(II.3)} \qquad x_{ik}(0) = \begin{cases} x_i^0 & \text{if } k = 0 \\ 0 & \text{if } k \geq 1\,. \end{cases}$$

We can easily verify that the system (II.2)–(II.3) is equivalent to the system (I.1). Since the functions $f_i(t)$ and $g_{ik}(x_i)$ are supposed to be sufficiently smooth, Eqs. (II.2) admit a unique system of solutions $\{x_{ik}(t)\}$ which satisfy the initial conditions (II.2).

In the next section we shall consider the case $f_i(t) \equiv f_i = \text{const.}$ in which case the system (II.2) is autonomous.

III. The steady-state solutions of Eqs. (II.2) with $f_i(t) \equiv f_i = \textit{constant}$ are determined by the following $N(K+1)$ equations:

$$\text{(III.1)} \qquad \begin{cases} x_{i0} = \dfrac{1}{1 + \exp\left\{-f_i - \sum\limits_{j=1}^{N} c_{ij}\, x_{j0} - \sum\limits_{k=1}^{K} b_{ik}\, x_{ik}\right\}} \\ x_{ik} = \dfrac{1}{p_{ik}}\, g_{ik}(x_i) \qquad (i = 1, \cdots, N\,;\ k = 1, \cdots, K)\,. \end{cases}$$

Under certain conditions satisfied by the constants $a, b_{ik}, c_{ij}, f_i, p_{ik}$ and $|g'_{ik}[x_i]|$, Eqs. (III.1) admit a unique system of solutions, say $\{x^*_{ik}\}$, by virtue of the principle of contraction mappings.

Put

$$\text{(III.2)} \qquad x_{ik} = x^*_{ik} + X_{ik}\,.$$

Then we find

$$\text{(III.3)} \qquad \begin{cases} \dfrac{dX_{i0}}{dt} = -aX_{i0} + a\sum\limits_{j=1}^{N} c_{ij}\, h_{ij}\, X_{i0} + a\sum\limits_{k=1}^{N} b_{ik}\, h_{ik}\, X_{ik} + Q_i(X)\,, \\ \dfrac{dX_{ik}}{dt} = g'_{ik}(x^*_{i0})\, X_{i0} - p_{ik}\, X_{ik} + R_{ik}(X)\,, \end{cases}$$

where

$$h_{ij} = \frac{1}{1 + \exp\left\{-f_i - \sum_{j=1}^{N} c_{ij} x^*_{j0} - \sum_{k=1}^{K} b_{ik} x^*_{ik}\right\}} - \frac{1}{\left\{1 + \exp\left\{-f_i - \sum_{j=1}^{N} c_{ij} x^*_{j0} - \sum_{k=1}^{K} b_{ik} x^*_{ik}\right\}\right\}^2}, \tag{III.4}$$

and $Q_i(X)$ and $R_{ik}(X)$ are the remainders of the Taylor expansions containing the nonlinear terms of the relevant functions at $x_{ik} = x^*_{ik}$. The functions $Q_i(X)$ and $R_{ik}(X)$ are negligible for small X_{ik}'s.

The stability properties of the linear equations obtained from Eqs. (III.3) dropping out the nonlinear terms $Q_i(X)$ and $R_{ik}(X)$ can be investigated by the well known methods, such a Routh-Hurwitz criterion, etc. We omit the details of this analysis.

In the next section we give a numerical scheme to establish the solutions in the case where $f_i(t)$ is no longer restricted to be a constant.

IV. Consider Eqs. (I.1), and, to simplify our presentation, put

$$\text{(IV.1)} \quad \begin{cases} w_{ik} = \int_0^t g_{ik}[x_i(s)] \exp\{-p_{ik}(t-s)\}\, ds, \\ v_i = f_i + \sum_{k=1}^{K} b_{ik} w_{ik} + \sum_{j=1}^{N} c_{ij} x_j & (f_i = f_i(t), \quad c_{ii} = 0), \\ u_i = \frac{1}{1 + e^{-v_i}}, & (i = 1, 2, \cdots, N). \end{cases}$$

Then Eqs. (I.1) can be written in the following compact form

$$\frac{dx_i}{dt} = a[u_i - x_i], \qquad x_i(0) = x_i^0. \tag{IV.2}$$

We wish to evaluate the value of $x_i(t)$ by the Taylor series expansion using the single step with a variable step size and variable order approximation method.

To this end put

$$\text{(IV.3)} \quad \begin{aligned} x_{ir}^{(m)} &= \left.\frac{d^m x_i}{dt^m}\right|_{t=rh} && (r = 0, 1, \cdots, R;\ m = 0, 1, \cdots, M), \\ x_{ir}^{(0)} &= x_i^0 && (i = 1, 2, \cdots, N). \end{aligned}$$

Then we have

$$x_{i,r+1} = x_{ir} + \frac{h}{1!} x'_{ir} + \frac{h^2}{2!} x''_{ir} + \cdots + \frac{h^M}{M!} x_{ir}^{(M)}, \tag{IV.4}$$

by Taylor's theorem, where M is so chosen that

$$\left| \frac{h^{M+1}}{(M+1)!} x_{ir}^{(M+1)} \right| \leq \varepsilon , \tag{IV.5}$$

ε being an arbitrarily given positive number. Clearly, $x_{i,r+1}$ will be known with an error less then ε if $x_{ir}^{(m)}$ $(m = 0 , 1 , \cdots , M)$ are known. To compute $x_{ir}^{(m)}$ we must find $u_{ir}^{(m-1)}$ and $x_{ir}^{(m-1)}$, where $u_{ir}^{(m-1)}$ is defined as above. By (IV.2), we have

$$x_{ir}^{(m)} = [u_{ir}^{(m-1)} - x_{ir}^{(m-1)}] . \tag{IV.6}$$

Using the notations

$$[n] = \frac{d^n u_i}{dv_i^n} \quad , \quad (n) = \frac{d^n v_i}{dt^n} , \tag{IV.7}$$

we can easily express $d^r u_i/dt^r$ in terms of $[n]$ and (n), $n = 1 , 2 , \cdots , r$. Indeed, by successive differentiations and use of the classical chain rule, we find

$$\begin{aligned}
\frac{du_i}{dt} &= [1]\,(1) , \\
\frac{d^2 u_i}{dt^2} &= [2]\,(1)^2 + [1]\,(2) , \\
\frac{d^3 u_i}{dt^3} &= [3]\,(1)^3 + 3\,[2]\,(1)\,(2) + [1]\,(3) , \\
\frac{d^4 u_i}{dt^4} &= [4]\,(1)^4 + 6\,[3]\,(1)^2\,(2) + 4\,[2]\,(1)\,(3) + 3\,[2]\,(2)^2 + [1]\,(4) , \\
\frac{d^5 u_i}{dt^5} &= [5]\,(1)^5 + 10[4]\,(1)^3\,(2) + 10\,[3]\,(1)^2\,(3) + 15\,[3]\,(1)\,(2)^2 + \\
&\quad + 5\,[2]\,(1)\,(4) + 10\,[2]\,(2)\,(3) + [1]\,(5) , \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{IV.8}$$

A general formula for $d^n u_i/dt^n$ is given in [3].

In [3] it has been shown that

$$\frac{d^r u_i}{dv_i^r} = u_i z_i P_r(z_i) , \tag{IV.9}$$

where $P_r(\theta)$ is a polynomial of degree $r - 1$ defined by the recurrence formula

$$P_1(\theta) = 1 , P_r(\theta) = (2\theta - 1) P_{r-1}(\theta) + \theta(\theta - 1) P'_{r-1}(\theta) \quad (r = 1 , 2 , 3 , \cdots) \tag{IV.10}$$

and

$$z_i = u_i e^{-v_i} . \tag{IV.11}$$

On the other hand we have

(IV.12)
$$\frac{d^r v_i}{dt^r} = \frac{d^r f_i}{dt^r} + \sum_{k=1}^{K} b_{ik} \frac{d^r w_{ik}}{dt^r} + \sum_{j=1}^{N} c_{ij} \frac{d^r x_i}{dt^r}$$

and

(IV.13)
$$\frac{d^r w_{ik}}{dt^r} = \frac{d^{r-1} g_{ik}(x_i)}{dt^{r-1}} - p_{ik} \frac{d^{r-1} w_{ik}}{dt^{r-1}} ,$$

by virtue of Eqs. (IV.1). Clearly, the derivatives $\frac{d^r g_{ik}(x_i)}{dt^r}$'s can be computed by Eqs. (IV.8) putting $[n^*] = \frac{d^n g_{ik}(x_i)}{dx_i^n}$ and $(n^*) = \frac{d^n x_i}{dt^n}$, $n^* = 1, 2, \cdots, r$. Thus, we can easily calculate $\frac{d^n w_{ik}}{dt^n}$ at $x = rh$ for any $n = 1, 2, 3, \cdots$ by the recurrence formula (IV.13) if we know $g_{ik}(x_i)$ and w_{ik}, $x = rh$. Thus we have to evaluate $w_{ik}(rh)$. To this end put

(IV.14)
$$w_{ik;r} = \int_0^{rh} g_{ik}[x_i(s)] \exp\{-p_{ik}(rh - s)\} ds .$$

By use of a suitable quadratic approximation of $g_{ik}(x_i(s))$ in $(r-1)h \leq s \leq rh$, we can show, by a not very long computation, that

(IV.15)
$$w_{ik;r} = e^{-p_{ik}h}\left[w_{ik;r-1} - \frac{1}{p_{ik}}\left(g_{ik;r-1} - \frac{1}{p_{ik}} g'_{ik;r-1} + \frac{2\gamma}{p_{ik}^2}\right)\right] + \frac{1}{p_{ik}}\left[g_{ik;r} - \frac{1}{p_{ik}} g'_{ik;r} + \frac{2\gamma}{p_{ik}^2}\right],$$

where

(IV.16)
$$g_{ik;r} = g_{ik}[x_i(rh)] \quad , \quad g'_{ik;r} = \left.\frac{dg_{ik}[x_i(s)]}{ds}\right|_{s=rh}$$

and

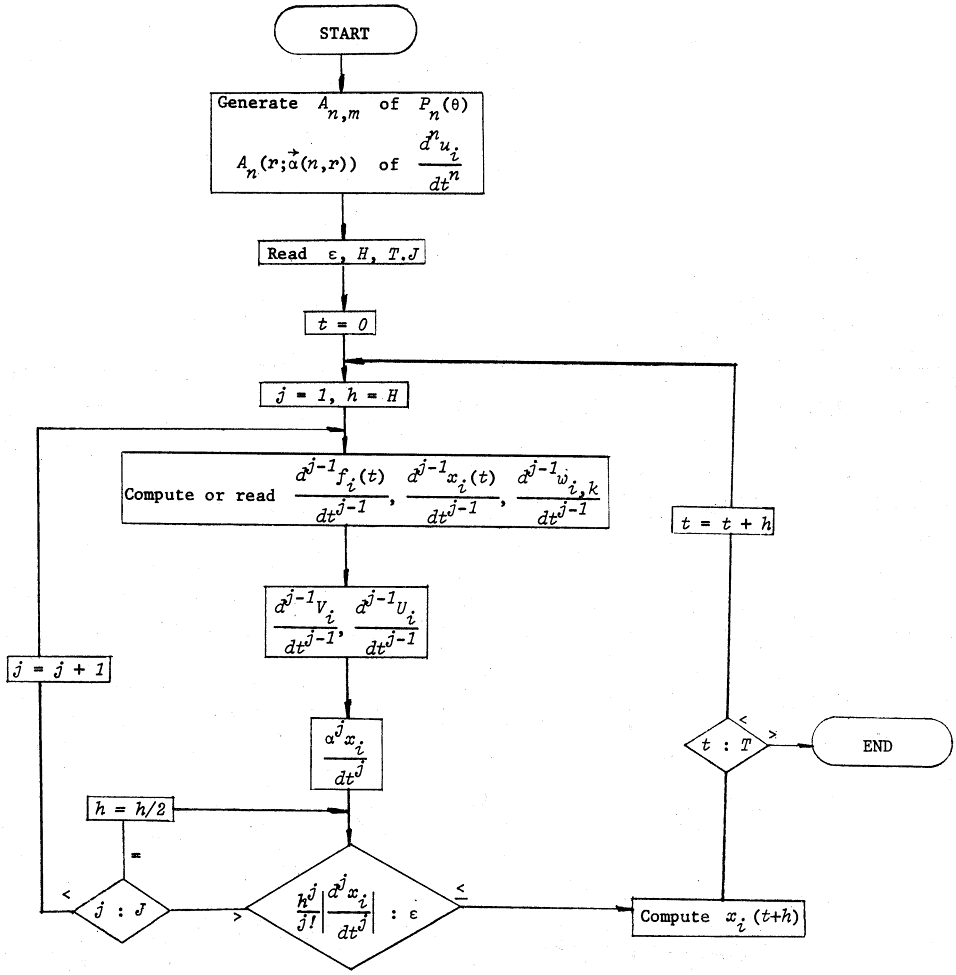
(IV.17)
$$\gamma = \frac{1}{h^2}[g_{ik;r} - g_{ik;r-1} - hg'_{ik;r-1}] .$$

Combining the formulas we obtain a recurrence relation connecting $x_{i,r+1}$ and x_{ir}. The flowchart for this combination is given below.

Using a high speed computer we can establish the solution of Eqs. (I.1) numerically.

Using a FORTRAN programming for IBM 360 computer we obtained solutions in approximately 2 minutes with $h = 10^{-4}$ and $R = 2000$.

FLOWCHART



V. The authors gratefully acknowledge the use of the computing facilities of the University of Alberta, and are pleased to emphasize once more that the invaluable research work of Professor M. Picone [5] and his Institute (INAC), as well as Vito Volterra's famous *Mathematical Theory of the Struggle for Life* [6], constitute a permanent example of how to embark upon, and how to solve, difficult problems arising from the study of natural phenomena.

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