Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Remarks on some representation theorems for convolution transforms

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.4, p. 562–566.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_4_562_0>

Analisi funzionale. — *Remarks on some representation theorems for convolution transforms.* Nota di DANY LEVIATAN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dà due criteri necessari e sufficienti perché una funzione $f(x)$ possa essere rappresentata da una trasformazione di convuluzione

$$f(x)=\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)\,, \qquad -\infty<x<\infty\,,$$

nel caso che il nucleo $G(x)$ sia della classe 1 o della classe 3 ed esista un x_0 tale che

$$\int_{-\infty}^{\infty} G(x_0-t)\,|d\alpha(t)|\leq M<\infty,$$

essendo $\alpha(t)$ una funzione localmente a variazione limitata.

1. INTRODUCTION

Recently Z. Ditzian [1] has discussed the problem of finding necessary and sufficient conditions on a function $f(x)$ so that it be represented as a convolution transform

$$f(x)=\int_{-\infty}^{\infty} G(x-t)\,d\alpha(t)\,, \qquad -\infty<x<\infty\,, \tag{1}$$

where $G(t)$ is a kernel belonging to Class I and where $\alpha(t)$ is locally of bounded variation and such that for some x_0

$$\int_{-\infty}^{\infty} G(x_0-t)\,|d\alpha(t)|\leq M<\infty\,. \tag{2}$$

This problem is interesting since for the first time in the representation theory for the convolution transform it allows $\alpha(t)$ to behave quite badly and restricts it via a condition that depends on the kernel $G(t)$. All the known results up until now (see [2], Ch. VII) have assumed $\alpha(t)$ to behave well. On the other hand the proof of the major result [1, Th. 3.1] is long and tedious and the purpose of this note is to replace [1] (3.1) by another condition (3) below, and shorten the proof considerably. Since the conditions in [1, Th. 3.1]

(*) Nella seduta del 14 aprile 1973.

and in our Th. 1 are both necessary and sufficient they are equivalent. However we do not know of a direct proof of this equivalence. Had we known it would presumably have shortened the proof of [1, Th. 3.1].

The advantage of our necessary and sufficient conditions is that they show the way to obtaining similar results for kernels $G(t)$ in Class III while Ditzian's result is restricted to Class I.

The reader is referred to the book of I. I. Hirschman and D.V. Widder [2] for the notation used in this paper.

2. MAIN RESULTS

The convolution kernels $G(t)$ we will be interested in will be kernels which satisfy

$$\int_{-\infty}^{\infty} e^{-st} G(t) \, dt = [E(s)]^{-1}$$

where

$$E(s) = e^{bs} \prod_{k=1}^{\infty} \left(1 - \frac{s}{a_k}\right) e^{s/a_k}$$

b and a_k are real and $\sum_{k=1}^{\infty} a_k^{-2} < \infty$.

$G(t)$ belongs to Class I if there are both positive and negative a_k's; it belongs to Class II if all the a_k's are positive and $\sum_{k=1}^{\infty} a_k^{-1} = \infty$; and it belongs to Class III if all the a_k's are positive and $\sum_{k=1}^{\infty} a_k^{-1} < \infty$.

Although the results in each class are analogous, different formulation is needed in each case and therefore we deal with each class separately.

$$\text{Denote} \quad \alpha_1 = \max_{a_k < 0} [a_k, -\infty] \qquad \text{and} \quad \alpha_2 = \min_{a_k > 0} [a_k, \infty].$$

Our first result is the one referred to in the introduction, the counterpart of [1, Th. 3.1].

THEOREM 1. *Let $G(t)$ belong to Class I. Then necessary and sufficient conditions in order that $f(x)$ possess the representation (1) for $-\infty < x < \infty$, where $\alpha(t)$ is locally of bounded variation and such that (2) holds for some x_0, are that $f \in C^{\infty}(-\infty, \infty)$, $f^{(k)}(x) = o(e^{\alpha_1 x})$, $x \to -\infty$, $f^{(k)}(x) = o(e^{\alpha_2 x})$ $x \to \infty$ for each $k = 0, 1, 2, \cdots$ and*

$$\int_{-\infty}^{\infty} H_n(x_0 - t) \, | P_n(D) f(t) | \, dt \leq M \qquad n = 0, 1, 2, \cdots. \tag{3}$$

Proof. First we prove necessity. Suppose $f(x)$ is represented by (1) where $\alpha(t)$ is locally of bounded variation and such that (2) holds for some x_0.

Then since G $\in$ Class I it follows by [2, Th. VI.5 1 a] that $f \in C^{\infty}(-\infty, \infty)$ and by [2, Th. VII.2.1] that $f^{(k)}(x) = o(e^{\alpha_1 x})$, $x \to -\infty$, $f^{(k)}(x) = o(e^{\alpha_2 x})$ $x \to \infty$ for each $k = 0, 1, 2, \cdots$ Furthermore by [2, Th. VI.5.1 a].

$$P_n(D) f(x) = \int_{-\infty}^{\infty} G_n(x-t)\, d\alpha(t) \qquad -\infty < x < \infty, \qquad n = 1, 2, 3, \cdots$$

Hence by Tonelli's theorem

$$\begin{aligned} &\int_{-\infty}^{\infty} H_n(x_0 - t)\, |P_n(D) f(t)|\, dt \\ &\leq \int_{-\infty}^{\infty} H_n(x_0 - t) \int_{-\infty}^{\infty} G_n(t-u)\, |d\alpha(u)|\, dt \\ &= \int_{-\infty}^{\infty} |d\alpha(u)| \int_{-\infty}^{\infty} H_n(x_0 - t)\, G_n(t-u)\, dt \\ &= \int_{-\infty}^{\infty} G(x_0 - u)\, |d\alpha(u)| \leq M, \qquad n = 1, 2, \cdots. \end{aligned}$$

This concludes the proof of the necessity.

Conversely by the conditions on f it follows that (see [2, Section VII.5])

$$f(x) = \int_{-\infty}^{\infty} H_n(x-t)\, P_n(D) f(t)\, dt, \qquad -\infty < x < \infty.$$

Hence

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \frac{H_n(x-t)}{H_n(x_0-t)}\, H_n(x_0 - t)\, P_n(D) f(t)\, dt \\ &= \int_{-\infty}^{\infty} \psi_n(x, t)\, d\varphi_n(t), \qquad -\infty < x < \infty, \end{aligned}$$

where

$$\varphi_n(t) = \int_0^t H_n(x_0 - u)\, P_n(D) f(u)\, du$$

and

$$\psi_n(x, t) = H_n(x-t) / H_n(x_0 - t).$$

By (3) the functions $\{\varphi_n\}$ are of uniformly bounded variations and are uniformly bounded in $(-\infty, \infty)$. Set $\psi(x, t) = G(x-t)/G(x_0-t)$, then (see [2, p. 160]) there exists a sequence $\{n_i\}$ such that for some φ of bounded variation

$$\lim_{i\to\infty} \int_{-\infty}^{\infty} \psi(x, t)\, d\varphi_{n_i}(t) = p_1 e^{\alpha_1(x-x_0)} + p_2 e^{\alpha_2(x-x_0)} + \int_{-\infty}^{\infty} \psi(x, t)\, d\varphi(t)$$

where p_1 and p_2 are constants.

Now

$$\left| f(x) - \int_{-\infty}^{\infty} \psi(x, t)\, d\varphi_{n_i}(t) \right|$$

$$= \left| \int_{-\infty}^{\infty} [\psi_{n_i}(x, t) - \psi(x, t)]\, d\varphi_{n_i}(t) \right|$$

$$\leq M \, \|\psi_{n_i}(x, t) - \psi(x, t)\|_{\infty} \to 0, \qquad \text{as} \quad i \to \infty .$$

(See [2, p. 160]).

Therefore

$$f(x) = p_1 e^{\alpha_1(x-x_0)} + p_2 e^{\alpha_2(x-x_0)} + \int_{-\infty}^{\infty} \psi(x, t)\, d\varphi(t)$$

which in turn implies

$$f(x) = p_1 e^{\alpha_1(x-x_0)} + p_2 e^{\alpha_2(x-x_0)} + \int_{-\infty}^{\infty} G(x-t)\, d\alpha(t) \tag{4}$$

where

$$\alpha(t) = \int_0^t \frac{d\varphi(u)}{G(x_0-u)} .$$

Now since $f(x) = o(e^{\alpha_1 x})$, $x \to -\infty$, and $\int_{-\infty}^{\infty} G(x-t)\, d\alpha(t) = o(e^{\alpha_1 x})$ $x \to -\infty$ (see [2, Th. VII.2.1]) it follows that $p_1 = 0$. Similarly $p_2 = 0$ and the proof follows by (4).

We have a similar result for $G \in$ Class III the proof of which is left to the reader.

THEOREM 2. *Let* $G(t)$ *belong to Class III and let* T *and* x_0, $-\infty \leq \leq T + b + \sum_{k=1}^{\infty} \frac{1}{a_k} < x_0 < \infty$ *be fixed. Then necessary and sufficient conditions* *in order that* $f(x)$ *possess the representation* (1) *for* $x > T + b + \sum_{k=1}^{\infty} \frac{1}{a_k}$ *where* $\alpha(t)$ *is locally of bounded variation in* (T, ∞) *and such that* (2) *holds are that* $f \in C^{\infty}(T, \infty)$, $f^{(k)}(x) = o(e^{\alpha_2 x})$, $x \to \infty$, *for each* $k = 0, 1, 2, \cdots$ *and* (3) *holds.*

References

[1] Z. Ditzian, *On representation of functions as convolution transforms*, « Annali di Matematica Pura ed Appl. », ser. 4, *92*, 95–106 (1972).

[2] I. I. Hirschman and D. V. Widder, *The convolution transform*, Princeton Univ. Press, Princeton 1955.