Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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Fixed point Theorems for a sum of nonlinear operators

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.4, p. 558–561.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_4_558_0>

Analisi funzionale. — *Fixed point Theorems for a sum of non linear operators.* Nota di S. P. SINGH (*), presentata (**) dal Corrisp. G. FICHERA.

RIASSUNTO. — In questa Nota viene fornito un miglioramento di un teorema di Krasnoselsky.

In recent years many authors have proved various fixed point theorems for non linear operators in Banach and Hilbert spaces. If C is a closed, bounded convex subset of a Banach space X, then a contraction mapping T, $\|Tx - Ty\| \leq k \|x - y\|$, $0 < k < 1$, for all x, y in C has a unique fixed point in C, but a non-expansive mapping, $\|Tx - Ty\| \leq \|x - y\|$, need not. In case X is uniformly convex Banach space [2] or more generally X is a reflexive Banach space with normal structure [6] then a non-expansive mapping has a fixed point in C.

Krasnoselsky [7] has proved that if C is a non-empty closed bounded and convex subset of a Banach space X and if $A : C \rightarrow X$, and $B : C \rightarrow X$ such that $Ax + By \in C$ for all x, y in C, furthermore, if A is a contraction mapping and B is compact (i.e. it is continuous and maps bounded sets into relatively compact sets), then $A + B$ has a fixed point in C.

A mapping $T : C \rightarrow X$ is called completely continuous if it maps weakly convergent sequences into strongly convergent sequences. If A is non-expansive, and B is completely continuous, and X is a Hilbert space then the above theorem has been proved in [4] and [14].

By putting the weaker condition $Ax + Bx \in C$ in place of $Ax + By \in C$ the above theorem has been given in [5], [8] and [11].

The aim of this Note is to prove a few fixed point theorems by using this weaker condition.

PRELIMINARY DEFINITIONS AND RESULTS

Let X be a Banach space and A be a bounded subset of X. The measure of non-compactness of A, denoted $\alpha(A)$, is defined as $\inf\{\varepsilon > 0 / A$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$, [9]. Using the concept of the measure of non-compactness k-set contractions have been defined [9] in the following way: If A is a subset of X and T a continuous mapping of A into X, then T is called a k–set contraction if for any given bounded set G in A then $\alpha(T(G)) \leq k\, \alpha(G)$ for some $k \geq 0$. The sum of

(*) This work was done while the Author was a fellow of the Summer Research Institute of the Canadian Mathematical Congress, Kingston.

(**) Nella seduta del 14 aprile 1973.

two k–set contraction map is a k–set contraction [10]. In case $k = 1$, we say 1–set contraction. A non-expansive mapping is clearly a 1–set contraction.

A continuous mapping $T: X \to X$ is called densifying, if for every bounded subset A of X such that $\alpha(A) > 0$, we have $\alpha(T(A)) < \alpha(A)$, [5].

Sadovsky [11] has introduced a condensing map in the following way: $\chi(T(A)) < \chi(A)$ for all $A \subset X$ such that $\chi(A) > 0$. Here $\chi(A)$ denotes the infimum of all real numbers $\varepsilon > 0$ such that A admits a finite ε–net.

THEOREM. (Sadovsky [11], Furi and Vignoli [5]). *Let* C *be a non-empty closed bounded and convex subset of a Banach space* X, *and let* $T: C \to C$ *be condensing (densifying) mapping. Then* T *has at least one fixed point in* C.

A contraction mapping is densifying and so is a compact mapping.

LEMMA 1. *Let* X *be a reflexive Banach space and* C *be a closed convex subset of* X. *Then every completely continuous operator* $T: C \to X$ *is also compact* [3, page 130].

THEOREM 1. *Let* X *be a reflexive Banach space and let* A *and* B *be two mappings of* C *into* X, *where* C *is a non-empty, closed bounded and convex subset of* X, *such that* .

1) A *is non-expansive and* $(1 - A)$ *is demi closed, and*
2) B *is completely continuous.*

Then there exists some $x_0 \in C$ *such that*

$$Ax_0 + Bx_0 = x_0 .$$

Proof. The proof is basically on the same lines as in [12]. Let k be a fixed positive number < 1, then the mapping kA is contraction and hence $kA + kB$ has a fixed point $x_k \in C$ by Sadovsky's theorem. By Lemma 1, $kA + kB$ is a condensing map. Let k_n be a sequence of numbers such that $0 < k_n < 1$, and $k_n \to 1$. Let $\{x_{k_n}\}$ be a sequence of points such that $k_n Ax_{k_n} + k_n Bx_{k_n} = x_{k_n}$, $x_{k_n} \in C$. Since X is reflexive and $\{x_{k_n}\}$ bounded, the sequence has a convergent subsequence $\{x_{k_{n_i}}\}$ converging to $x_0 \in C$ weakly. We claim that $Ax_0 + Bx_0 = x_0$. Now,

$$x_{k_{n_i}} - Ax_{k_{n_i}} = -\left(\frac{1 - k_{n_i}}{k_{n_i}}\right) x_{k_{n_i}} + \frac{x_{k_{n_i}}}{k_{n_i}} - Ax_{k_{n_i}} = -\left(\frac{1 - k_{n_i}}{k_{n_i}}\right) x_{k_{n_i}} + Bx_{k_{n_i}}$$

since $\{x_{k_{n_i}}\}$ is bounded and B is completely continuous, we get $(x_{k_{n_i}} - Ax_{k_{n_i}}) \to$ $\to Bx_0$. Since $(1 - A)$ is demi closed we get that $(1 - A)x_0 = Bx_0$, i.e. $Ax_0 + Bx_0 = x_0$.

COROLLARY 1. *As a corollary we have a theorem due to Srinivasacharyulu* [12], *where it has been assumed that* $Ax + By \in C$ *for all* $x, y \in C$.

The following result is due to Zabreiko, Kachurovsky and Krasnoselsky [14].

COROLLARY 2. *Let* C *be a closed, bounded and convex subset of a Hilbert space* H. *Let* $T: C \to C$ *be a non linear operator such that* $T = A + B$, *where* A *is non-expansive and* B *is completely continuous. Then* T *has at least one fixed point.*

Proof. A Hilbert space is a reflexive Banach space and therefore the result follows from Theorem 1, as $1 - A$ is always demi closed [3, page 51].

COROLLARY 3. *Let* C *be a non-empty, closed bounded and convex subset of a Hilbert space* H *and let* $T : C \to H$ *be a non linear operator such that* $T = A + B$ *where*

1) $Ax + By \in C$ *for* $x, y \in C$;
2) A *is non-expansive and*
3) B *is completely continuous.*

Then T has a fixed point. This theorem is due to Edmunds [4].

THEOREM 2. *Let* X *and* C *be the same as in Theorem 1. If* A *is a 1-set contraction and* $(1 - T)$ *is demi closed and* B *is completely continuous, then* $T = A + B$ *has a fixed point in* C.

Proof. If $k < 1$, then kA is a condensing map and $kA + kB$ is also a condensing map, therefore we can apply a theorem of Sadovsky and the remaining part follows as in Theorem 1.

COROLLARY 1. *Since a non-expansive mapping is a 1-set contraction, Theorem 1, becomes a corollary of Theorem 2.*

THEOREM 3. *Let* C *be a non-empty, weakly compact convex subset of a Banach space* X. *If* $T : C \to C$ *is a non linear operator such that*

$$T = A + B,$$

where $A : C \to C$ *is non-expansive and* $B : C \to C$ *is compact, and* $1 - T$ *is convex. Furthermore, if* $\inf \| x - Tx \| = 0$ *then* T *has a fixed point.*

Proof. Since C is a weakly compact space and $\| x - Tx \|$ is weakly lower semi continuous on C therefore $\| x - Tx \|$ has its infimum on C. i.e. there exists $x_0 \in C$ such that

$$\| (1 - T) x_0 \| = \inf_{x \in C} \| (1 - T) x \| .$$

Since

$$\inf_{x \in C} \| (1 - T) x \| = 0 ,$$

therefore $Tx_0 = x_0$. i.e. T has a fixed point.

COROLLARY 1. Srinivasacharyulu [13]. *Let* C *be a non-empty bounded closed convex set containing the origin as interior point in a reflexive Banach space* X, *and let* A *be a non-expansive mapping in* C, *let* $B : C \to C$ *be completely continuous and* $Ax + By \in C$ *for all* $x, y \in C$. *If* $(1 - A - B)$ *is convex on* C, *then there exists at least one* $x_0 \in C$ *such that* $Ax_0 + Bx_0 = x_0$.

Proof. Since a bounded subset of a reflexive Banach space is weakly compact therefore C is weakly compact. Also, $\| x - Tx \| = \| x - Ax - Bx \|$ is weakly lower semi continuous on C, since a convex continuous real valued function on a Banach space is weakly lower semi continuous. Therefore $\| x - Tx \|$ has its infimum on C i.e. there exists $x_0 \in C$ such that

$$\| (1 - T) x_0 \| = \inf_{x \in C} \| (1 - T) x \| .$$

We need to show that $\inf_{x \in C} \| (1 - T) x \| = 0$. Consider kT, where $0 < k < 1$. Then since C is convex, $kTx \in C$ for all $x \in C$. Thus, there exists a point $x_k \in C$ such that $kTx_k = x_k$ by a theorem of Sadovsky. Let k_n be a sequence of numbers $0 < k_n < 1$ such that $k_n \to 1$. Then

$$x_k - Ax_k - Bx_k = (k - 1)(Ax_k + Bx_k) .$$

Since T maps bounded sets into bounded sets we have

$$\| Tx_k \| \leq K ,$$

and therefore

$$\| x_{k_n} - Ax_{k_n} - Bx_{k_n} \| \leq (k_n - 1) K \to 0 .$$

This implies that

$$\inf_{x \in C} \| x - Ax - Bx \| = 0 .$$

COROLLARY 2. *If we take* $B = 0$, *then we get a result due to Belluce and Kirk* [1].

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