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Cesàro absolutely p-summing operators

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Analisi funzionale. — *Cesàro absolutely p–summing operators.* Nota di GHEORGHE CONSTANTIN, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si introduce una classe di operatori assolutamente p–sommabili di Cesàro e se ne dànno alcune proprietà.

1. In [1] is introduced the Cesàro-Hilbert-Schmidt operator which generalizes the Hilbert-Schmidt operator. It is known [3], [4] that, in Hilbert spaces, the absolutely p–summing operators, $(1 \leq p < \infty)$, introduced by A. Pietsch in [4], coincide with the class of all Hilbert-Schmidt operators.

In this Note we introduce the class of Cesàro absolutely p–summing operators and we give some properties for this class of operators.

2. DEFINITION 2.1. *Let* E *and* F *be normed spaces. An operator* $T: E \rightarrow F$ *is called a Cesàro absolutely p–summing operator* $1 \leq p < \infty$, *if for all sequence* $\{x_n\} \subset E$ *there exists a constant* $C > 0$ *such that*

$$\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \| Tx_k \|\right)^p\right]^{1/p} \leq C \sup_{\| a \| \leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} | \langle x_k , a\rangle |\right)^p\right]^{1/p}.$$

The smallest constant C such that the above inequality is satisfied will be denoted by $\nu_p(T)$ and we have

PROPOSITION 2.1. *The set of Cesàro absolutely p–summing operators is a normed space under the norm* ν_p.

The proof is the same as for Cesàro-Hilbert-Schmidt operators [1].

PROPOSITION 2.2. *Every Cesàro absolutely p–summing operator* T *is bounded and* $\| T \| \leq \nu_p(T)$.

PROPOSITION 2.3. *If* F *is a Banach space then the set of Cesàro absolutely p–summing operators is a Banach space under the norm* ν_p.

Proof. Let $\{T_n\}$ be a Cauchy sequence. Since

$$\| T_s - T_q \| \leq \nu_p(T_s - T_q)$$

it follows that $\{T_n\}$ is a Cauchy sequence in the Banach space $\mathfrak{L}(E, F)$ and hence there exists a bounded linear operator T with the property that $\lim_{s \to \infty} \| T - T_s \| = 0$. But for each $\varepsilon > 0$ there exists a number $n_0(\varepsilon)$ such that

$$\nu_p(T_s - T_q) \leq \varepsilon \quad , \quad \forall s , q > n_0(\varepsilon)$$

and hence

$$\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \| (T_s - T_q) x_k \|\right)^p\right]^{1/p} \leq \varepsilon \sup_{\| a \| \leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} | \langle x_k , a\rangle |\right)^p\right]^{1/p}$$

(*) Nella seduta del 14 aprile 1973.

for all $s, q > n_0(\varepsilon)$. If $s \to \infty$ we obtain

$$\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \|(T - T_q)\, x_k\|\right)^p\right]^{1/p} \leq \varepsilon \sup_{\|a\|\leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} |\langle x_k , a\rangle|\right)^p\right]^{1/p}$$

for all $q > n_0(\varepsilon)$. It follows that

$$\nu_p(T - T_q) \leq \varepsilon \quad , \quad \forall q > n_0(\varepsilon)$$

and hence T is a Cesàro absolutely p–summing operator which is at the same time the limit of the Cauchy sequence $\{T_n\}$ under the norm ν_p.

PROPOSITION 2.4. *Let* E, F, G *be normed linear spaces. If* $T \in \mathfrak{L}(E, F)$ *and* $S : F \to G$ *is a Cesàro absolutely* p*–summing operator, then* ST *is a Cesàro absolutely* p*–summing operator and*

$$\nu_p(ST) \leq \|T\|\, \nu_p(S) .$$

If $T : E \to F$ *is a Cesàro absolutely* p*–summing operator and* $S \in \mathfrak{L}(F, G)$, *then* ST *is a Cesàro absolutely* p*–summing operator and*

$$\nu_p(ST) \leq \|S\|\, \nu_p(T) .$$

Proof. Let $\{x_k\}$ a sequence of elements of E, then

$$\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \|STx_k\|\right)^p\right]^{1/p} \leq \nu_p(S) \sup_{\|a\|\leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} |\langle Tx_k , a\rangle|\right)^p\right]^{1/p} =$$

$$= \nu_p(S)\,\|T\| \sup_{\|a\|\leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \left|\left\langle x_k , \frac{T^*}{\|T\|}\, a\right\rangle\right|\right)^p\right]^{1/p} \leq$$

$$\leq \nu_p(S)\,\|T\| \sup_{\|b\|\leq 1} \left[\sum_{k=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} |\langle x_k , b\rangle|\right)^p\right]^{1/p}$$

and the first part is proved.

For the last part, we observe that

$$\left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k-1}^{n} \|TSx_k\|\right)^p\right]^{1/p} \leq \|S\| \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \|Tx_k\|\right)^p\right]^{1/p} \leq$$

$$\leq \|S\|\, \nu_p(T) \sup_{\|a\|\leq 1} \left[\sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} |\langle x_k , a\rangle|\right)^p\right]^{1/p}$$

and the proposition is proved.

The above properties suggest a similarity with the compact operators. Moreover, if E and F are Hilbert spaces we have

PROPOSITION 2.5. *Every Cesàro absolutely* p*–summing operator, for* $p \geq 2$ *is a compact operator.*

Proof. We use the Pelczyński's device [3]. For every orthonormal sequence $\{e_n\} \subset E$ we have that $\|Te_k\| \to 0$ for $k \to \infty$.

If this is not so, then there exists $\varepsilon > 0$ and an orthonormal sequence $\{e_n\} \subset E$ such that

$$\| Te_n \| > \varepsilon , \qquad n = 1, 2, 3, \cdots$$

In this case we obtain

$$\left[\sum_{n=1}^{N} \frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n} \| Te_k \|\right)^p\right]^{1/p} \geq \varepsilon\left(\sum_{n=1}^{N}\frac{1}{n}\right)^{1/p}, \qquad N = 1, 2, 3, \cdots$$

On the other hand, from the fact that $\{e_n\}$ is an orthonormal sequence, it follows that

$$\left[\sum_{k=1}^{N} |\langle e_k, a\rangle|^2\right]^{1/2} \leq \| a \| \qquad \text{for} \quad a \in E .$$

Since $p \geq 2$, we have

$$\sup_{\|a\|\leq 1}\left[\sum_{n=1}^{N} |\langle e_n, a\rangle|^p\right]^{1/p} \leq \sup_{\|a\|\leq 1}\left[\sum_{n=1}^{N} |\langle e_n, a\rangle|^2\right]^{1/2}$$

and therefore

$$\sup_{\|a\|\leq 1}\left[\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\sum_{k=1}^{n}|\langle e_k, a\rangle|\right)^p\right]^{1/p} \leq \sup_{\|a\|\leq 1}\left[\sum_{n=1}^{N}\frac{1}{n}\left(\frac{n^{p-1}}{n^p}\sum_{k=1}^{n}|\langle e_k, a\rangle|^p\right)\right]^{1/p} =$$

$$= \sup_{\|a\|\leq 1}\left[\sum_{n=1}^{N}\frac{1}{n^2}\left(\sum_{k=1}^{n}|\langle e_k, a\rangle|^p\right)\right]^{1/p} \leq \sup_{\|a\|\leq 1}\left[\sum_{n=1}^{N}\frac{1}{n^2}\|a\|^2\right]^{1/p} = \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{1/p} .$$

Substituting these quantities in the relation of definition of the operator T, we obtain

$$\varepsilon\left(\sum_{n=1}^{N}\frac{1}{n}\right) \leq \left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{1/p}$$

a contradiction which proves the proposition.

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