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Iterative solution of linear operator equations in Banach spaces

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Equazioni lineari in uno spazio di Banach. — *Iterative solution of linear operator equations in Banach spaces.* Nota di SIMEON REICH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Un recente teorema sulla media ergodica è usato per l'approssimazione delle soluzioni dell'equazione $y - Vy \leq z$ dove z appartiene ad uno spazio di Banach E e $V: E \to E$ continua e lineare.

Let E be a Banach space and $V: \to E$ continuous and linear. In this note we consider the iterative construction of approximate solutions of linear operator equations of the type

$$(I - V)\, y = z \tag{1}$$

where I denotes the identity operator and $z \in E$ is given. We intend to state and prove a general theorem which extends many of the results which have been recently established by several Authors. The simple and transparent proof is based on a generalization of the mean ergodic theorem obtained by Leviatan and Ramanujan in [11]. Thus we avoid the more complicated arguments employed by Dotson [3, 4, 5, 6] and Groetsch [8] who used Eberlein's ergodic theorem [7, p. 220]. Moreover, we do not assume the powers of V to be uniformly bounded nor do we restrict our Toeplitz matrices to be lower-triangular, scalar-valued, and non-negative.

Let N denote the set of all non-negative integers. Let $A = \{A_{nk} : n, k \in N\}$ be a matrix each of its entries is a continuous linear self-mapping of E. A is said to be a Toeplitz matrix if for each convergent sequence $\{x_n : n \in N\}$ in E, the sequence $\{y_n : n \in N\}$ defined by $y_n = \sum_{k=0}^{\infty} A_{nk} x_k$ exists and converges to the limit of $\{x_n\}$. It is known [14, p. 18] that A is a Toeplitz matrix if and only if it satisfies the following three conditions.

$$\sup \left\{ \left\| \sum_{k=0}^{r} A_{nk} x_k \right\| : \quad \| x_k \| \leq 1 \quad (k \in N) \quad \text{and} \quad n, r \in N \right\} < \infty ; \tag{2}$$

$$\lim_{n \to \infty} A_{nk} x = 0 \qquad \text{for each} \quad k \in N \quad \text{and each} \quad x \in E ; \tag{3}$$

$$\sum_{k=0}^{\infty} A_{nk} x \qquad \text{exists for each} \quad n \in N \quad \text{and each} \quad x \in E , \tag{4}$$

$$\text{and} \quad \lim_{n \to \infty} \sum_{k=0}^{\infty} A_{nk} x = x .$$

We are now ready to quote the Leviatan-Ramanujan result.

(*) Nella seduta del 14 aprile 1973.

Mean Ergodic Theorem [11, p. 114]. Let E be a Banach space and $V : E \to E$ continuous and linear. Suppose that $A = \{A_{nk} : n , k \in N\}$ is a Toeplitz matrix of continuous linear operators on E and that V commutes with each A_{nk}. Let $\{q_n : n \in N\}$ be an increasing sequence of non-negative reals. Assume that

$$(5) \qquad T_n x = \sum_{k=0}^{\infty} A_{nk} V^k x \quad \text{exists for each} \quad x \in E \quad \text{and each} \quad n \in N ,$$

$$\text{and } \{T_n x : n \in N\} \quad \text{is weakly relatively compact;}$$

$$(6) \qquad \sum_{k=0}^{\infty} \| A_{nk} - A_{n,k+1} \| \, q_{k+1} \to 0 \qquad \text{as} \quad n \to \infty ;$$

$$(7) \qquad \| V^n \| \leq q_n \qquad \text{for each} \quad n \in N .$$

Then for each $x \in E$, $T_n x$ converges strongly to Px, where P is the linear continuous projection of E onto the null space of $I - V$ parallel to the closure of $R(I - V)$, the range of $I - V$.

In addition to the above-mentioned operators T_n we formally define the operators S_n $(n \in N)$ by

$$(8) \qquad S_n x = \sum_{k=1}^{\infty} A_{nk} \left(\sum_{j=0}^{k-1} V^j x \right)$$

where $x \in E$. Let x and z belong to E and consider the sequence $\{x_n : n \in N\}$ defined by

$$(9) \qquad x_n = T_n x + S_n z .$$

THEOREM. *Suppose that all the hypotheses of the Mean Ergodic Theorem hold.*

(*a*) *If* $z \in R(I - V)$, *then given any* $x \in E$, $\{x_n\}$ *exists and converges strongly to a point* y *which satisfies* (1).

(*b*) *Assume that* A *has the following property*:

(10) *For each* $x \neq 0$ *in* E, *either* $\sum_{k=0}^{\infty} k A_{nk} x$ *does not exist for all sufficiently large* n, *or* $\left\| \sum_{k=0}^{\infty} k A_{nk} x \right\| \to \infty$ *as* $n \to \infty$.

If for some $x \in E$, $\{x_n\}$ *exists and some subsequence* $\{x_m\} \subset \{x_n\}$ *is weakly convergent, then* $z \in R(I - V)$.

Proof.

(*a*) Let $w - Vw = z$. Then $S_n z = \sum_{k=0}^{\infty} A_{nk} w - T_n w$, so that $\{x_n\}$ exists and by the Mean Ergodic Theorem converges to $y = Px + w - Pw$. Furthermore, $y - Py = w - Pw = z$.

(*b*) Let $\{x_m\}$ converge weakly to y. The Mean Ergodic Theorem implies that $\{S_m z\}$ converges weakly to $y - Px$. It follows that $\{(I - V) S_m z\}$ converges weakly to $y - Vy$. But $(I - V) S_m z = \sum_{k=0}^{\infty} A_{mk} z - T_m z$. Hence $z = Pz + y - Vy$ and $S_m z = \sum_{k=0}^{\infty} k A_{mk} Pz + \sum_{k=0}^{\infty} A_{mk} y - T_m y$. An appeal to condition (10) yields $Pz = 0$ and this completes the proof.

Suppose that A is Toeplitz and scalar-valued. On the one hand, such matrices need not possess property (10). On the other hand, condition (10) does not imply total regularity [9, p. 54]. Of course, if A is non-negative, it is totally regular and therefore satisfies (10).

The following propositions can be immediately deduced from the Theorem: [1, Theorem 1], [2, Theorem 1], [4, Theorem 7], [5, Theorem 1], and [8, Theorem 2].

Euler's scalar-valued method $E_t\,(0 < t < 1)$ is defined by $a_{nk} = \binom{n}{k} t^k (1 - t)^{n-k}\,(0 \le k \le n)$, $a_{nj} = 0\ (j > n)$ where $n \in N$. The original Outlaw-Groetsch conjecture [13, p. 431] dealt with $t = 1/2$. In Dotson's affirmative solution [4, 5], $0 < t < 1$, $q_n = 1\ (n \in N)$ and E is uniformly convex. We believe that our approach gives a more illuminating answer. In particular, observe that $\{q_n\}$ can be $o(\sqrt{n})$ in Euler's method [12, p. 312]. Also we can now employ (inter alia) appropriate Hausdorff and Jakimovski's $[F, d_n]$ [10] methods. These methods contain Euler's method as a very special case. A step in the latter direction has been taken by Groetsch [8].

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