Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On the behavior of the solutions of $x'' + q(t)f(x) = r(t)$

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.4, p. 544–550.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_4_544_0>

Equazioni differenziali ordinarie. — *On the behavior of the solutions of $x''+q(t)f(x)=r(t)$.* Nota di JOHN R. GRAEF (*) e PAUL W. SPIKES (*), presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Si esamina il comportamento asintotico delle soluzioni nel caso che $\lim_{t\to\infty} r(t)/q(t)$ sia positivo e $r(t)/q(t)$ sia monotona.

1. INTRODUCTION

In [1] the Authors studied the asymptotic behavior of solutions of

$$x''+q(t)f(x)=r(t)$$

under the assumption that the quotient $r(t)/q(t)$ approaches zero monotonically as $t\to\infty$. In this paper we discuss the behavior of solutions of the above equation assuming that $r(t)/q(t)$ monotonically approaches a positive constant. We begin first with a boundedness result.

2. BOUNDEDNESS OF SOLUTIONS

We consider the equation

$$x''+q(t)f(x)=r(t) \tag{1}$$

where $q, r: [t_0, \infty) \to \mathrm{R}$, $f: \mathrm{R} \to \mathrm{R}$, q, r and f are continuous, $q(t)>0$, and $r(t)>0$. We define $\mathrm{F}(x)=\int_0^x f(s)\,ds$ and make the following assumptions about equation (1). We assume that

$$r'(t)\geq 0 \tag{2}$$

and that there are positive constants d, j, k and M such that

$$\mathrm{F}(x)\geq d\,|x|^{1+j} \qquad \text{for} \quad |x|\geq k, \tag{3}$$

$$\mathrm{H}(t)=r(t)/q(t) \qquad \text{increases monotonically to M as} \quad t\to\infty. \tag{4}$$

We note that, by Theorem 2.1 in [1], condition (3) guarantees that solutions of (1) are defined for all $t\geq t_0$.

It will often be convenient to write equation (1) as the system

$$x'=y \quad , \quad y'=-q(t)f(x)+r(t)\,. \tag{5}$$

(*) Supported in part by Mississippi State University Biological and Physical Sciences Research Institute.

(**) Nella seduta del 14 aprile 1973.

THEOREM 1. *Under conditions* (2)–(4) *all solutions of* (1) *are bounded.*

Proof. By (3), $F(x)$ is bounded from below so $F(x) \geq -K$ for some $K > 0$. Let $V(x, y, t) = y^2/2r(t) + (q(t)/r(t))(F(x) + K)$. Then $V' = y - y^2 r'(t)/2r^2(t) + (q(t)/r(t))'(F(x) + K) \leq y$. Integrating, we have $(q(t)/r(t))\,F(x(t)) \leq K_1 + x(t)$ for some $K_1 > 0$. Thus $F(x(t)) \leq (K_1 + x(t))M$. If $|x(t)| \geq k$, then $d\,|x(t)|^{1+j} \leq F(x(t)) \leq K_1 M + |x(t)|M$ so $x(t)$ is bounded.

COROLLARY 2. *Under conditions* (2)–(4), *if* $r(t)$ *is bounded, then all solutions of system* (5) *are bounded.*

Proof. From the proof of Theorem 1, $V(t)$ is bounded so $y^2(t)/2r(t) \leq B$ for some $B > 0$. Hence $y^2(t) \leq 2Br(t)$ so $y(t)$ is bounded and thus solutions of (5) are bounded.

In order to see that condition (3) is sharp, consider the equation

$$x'' + \tanh x = (t^2 - 1)/(t^2 + 1) - 1/t^2.$$

Now $F(x) = \ln(\cosh x)$ satisfies $|x| - \ln 2 < F(x) \leq |x|$ for all x. However $x(t) = \ln t$ is an unbounded solution of the above equation.

3. ASYMPTOTIC BEHAVIOR

The classification of solutions used in this part of the paper is the same as that used in Section 4 of [1]. Clearly the conclusion of Theorem 3 holds for Z-type solutions (have arbitrarily large zeros but do not change sign). A trivial modification of the proof of Theorem 4 shows that it also holds for Z-type solutions.

We shall make the following additional assumptions on equation (1). We assume that

$$q'(t) \geq 0 \tag{6}$$

$$q \in C^3 \quad \text{and} \quad q(t) \to \infty \quad \text{as} \quad t \to \infty \tag{7}$$

$$\int_{t_0}^{t} |(q^{-1}(s))'''|\,ds = o(\ln q(t)), \qquad t \to \infty \tag{8}$$

$$xf(x) > 0 \quad \text{if} \quad x \neq 0 \tag{9}$$

$$f'(x) \geq 0 \quad \text{for all } x \tag{10}$$

and that there are positive constants d and j such that

$$F(x) \geq d\,|x|^{1+j} \quad \text{for all} \quad x. \tag{11}$$

THEOREM 3. *Suppose conditions* (2), (4), *and* (6)–(11) *hold. Let* $x(t)$ *be an oscillatory solution of* (1) *and let* $h > 0$ *be given. Then there exists* $a > t_0$ *such that*:

(i) *If* $t \geq a$ *and* $x(t) < 0$, *then* $|x(t)| < h$.

(ii) *If for* $c > b > a$, $x(b) = x(c) = 0$ *and* $x(t) < 0$ *for* t *in* (b, c), *then* $c - b < h$.

Proof. Let $V(x, y, t) = y^2/2r(t) + q(t) F(x)/r(t)$; then $V' \leq y - y^2 r'(t)/2r^2(t)$. Integrating, we obtain

$$V(t) + \int_{t_0}^{t} [y^2(s) r'(s)/2r^2(s)] \, ds \leq V(t_0) + x(t) - x(t_0),$$

and since $x(t)$ is bounded by Theorem 1, we have

$$\int_{t_0}^{\infty} [y^2(s) r'(s)/2r^2(s)] \, ds < \infty.$$

Since $H(t) = r(t)/q(t)$ is increasing, $H'(t) \geq 0$ so $q'(t)/2q^2(t) \leq \leq M(r'(t)/2r^2(t))$. Thus

$$\int_{t_0}^{\infty} [y^2(s) q'(s)/2q^2(s)] \, ds < \infty. \tag{12}$$

Next we define

$$W_z(x, y, t) = y^2/2q(t) + F(x) - \int_{z}^{t} [r(s) y(s)/q(s)] \, ds$$

and we see that $W_z'(t) = -y^2 q'(t)/2q^2(t)$. Integrating, we have

$$W_z(t) = W_z(z) - \int_{z}^{t} [y^2(s) q'(s)/2q^2(s)] \, ds.$$

Let $\{t_n\}$ be a monotonically increasing sequence of zeros of $y(t)$ such that $t_n \to \infty$ as $n \to \infty$. Then

$$W_{t_n}(t) = F(x(t_n)) - \int_{t_n}^{t} [y^2(s) q'(s)/2q^2(s)] \, ds.$$

Hence,

$$d\,|x(t_n)|^{1+j} \leq W_{t_n}(t) + \int_{t_n}^{t} [y^2(s) q'(s)/2q^2(s)] \, ds. \tag{13}$$

Now let $x(t)$ be an oscillatory solution of (1) and suppose that (i) does not hold. Then there exists $h > 0$ such that for any $a > t_0$ there exists

$b \geq a$ with $x(b) \leq -h$. By (12) we can choose $a > t_0$ so that

$$\int_a^\infty [y^2(s)\, q'(s)/2\, q^2(s)]\, ds < dh^{1+j}/2 .$$

By Theorem 1, $x(t)$ is bounded, say $|x(t)| \leq B$ for some $B > 0$, so we let $m = Mh/4B$. Since $x(t)$ is oscillatory and $H(t)$ is increasing monotonically to M, there exists $b > a$ such that $x(b) = 0$ and for $t \geq b$ we have

$$(14) \qquad H(t) > M/2 \quad \text{and} \quad H(t) - H(b) < m .$$

There is a sequence $\{a_n\}$ increasing monotonically to ∞ with $a_1 > b$ and such that

$$(15) \qquad x(a_n) \leq -h \quad \text{and} \quad y(a_n) = 0 .$$

Observe next that $W'_{a_1}(t) \leq 0$ for $t > a_1$, and by (13), $W_{a_1}(t) \geq dh^{1+j}/2$ for $t > a_1$, so $W_{a_1}(t) \to L > 0$ as $t \to \infty$. Next we let $R(t) = q^{-1}(t)$ and $P(t) = 3\, W_{a_1}(t) + R''(t)\, x^2(t)/2 - R'(t)\, x(t)\, y(t)$. Then

$$P'(t) = R'''(t)\, x^2(t)/2 + q'(t)\, [-y^2(t)/2\, q(t) - f(x(t))\, x(t) + r(t)\, x(t)/q(t)]/q(t) .$$

From the proof of Theorem 4.5 in [1] we know that (10) implies $x(t) f(x(t)) \geq F(x(t))$ so we have

$$P'(t) \leq R'''(t)\, x^2(t)/2 + q'(t)\, [-y^2(t)/2\, q(t) - F(x(t)) + r(t)\, x(t)/q(t)]/q(t).$$

From a mean value theorem for integrals we have

$$\int_{a_1}^t H(s)\, y(s)\, ds = H(t)\, x(t) - H(a_1)\, x(a_1) - x(w)\, [H(t) - H(a_1)]$$

where $a_1 \leq w \leq t$. Hence

$$\int_{a_1}^t H(s)\, y(s)\, ds - H(t) x(t) \geq -|x(w)|\, [H(t) - H(a_1)] - H(a_1)\, x(a_1).$$

Now $|x(w)| \leq B$ and by (14), $0 \leq H(t) - H(a_1) < m$ and $H(a_1) > M/2$. Also, $x(a_1) \leq -h$ by (15). Thus we have

$$\int_{a_1}^t H(s)\, y(s)\, ds > H(t)\, x(t) \qquad \text{for} \quad t > a_1 .$$

Thus $P'(t) \leq R'''(t)\, x^2(t)/2 - q'(t)\, L/q(t)$. An integration of this inequality from a_1 to a_n, $n > 1$, yields

$$L \cdot \ln(q(a_n)) < C_0 + |R''(a_n)|\, B^2/2 + B^2 \int_{a_1}^{a_n} R'''(s)\, ds/2 .$$

But $|R''(a_n)| \leq \int_{a_1}^{a_n} |R'''(s)|\, ds + |R'''(a_1)|$ so

$$L < C_1/\ln(q(a_n)) + \left[B^2 \int_{a_1}^{a_n} |R'''(s)|\, ds\right]/\ln(q(a_n))$$

for each $n > 1$, which is impossible in light of conditions (7) and (8). This completes the proof of (i).

In order to prove (ii) we first note that it is easy to show, from the properties of $H(t)$, $F(x)$ and $W_a(x, y, t)$, that $y^2(t)/q(t)$ is bounded on $[a, \infty)$. And since we have that $q(t) \to \infty$ as $t \to \infty$, $y(t)/q(t) \to 0$ as $t \to \infty$. It then follows immediately from (4) that $y(t)/r(t) \to 0$ as $t \to \infty$. Now let $h > 0$ be given and choose a so that $|y(t)|/r(t) < h/2$ for $t > a$. Let $[b, c]$ be any interval such that $b > a$, $x(b) = x(c) = 0$, and $x(t) < 0$ for t in (b, c). Integrating equation (1) twice we have

$$x(t) = x'(b)(t-b) + \int_b^t \left(\int_b^w [r(s) - q(s) f(x(s))]\, ds \right) dw$$

$$= x'(b)(t-b) + \int_b^t (t-w)[r(w) - q(w) f(x(w))]\, dw.$$

For $t = c$ we obtain $x'(b)(c-b) + r(b)(c-b)^2/2 \leq 0$, so $(c-b)/2 \leq \leq |x'(b)|/r(b) < h/2$ and (ii) is proved.

THEOREM 4. *Suppose conditions* (2), (4) *and* (6)–(11) *hold. Let* $x(t)$ *be an oscillatory solution of* (1) *and let* $h > 0$ *be given. Then there exists* $T > t_0$ *such that* $-h < x(t) < (M/d + h)^{1/j}$ *for* $t > T$.

Proof. If $x(t)$ is an oscillatory solution of (1) and h is a positive number, then by Theorem 3 there exists $a > t_0$ such that if $t \geq a$ and $x(t) < 0$, then

$$x(t) > \max\{-h, -(dh/2M)(M/d)^{1/j}\} \tag{16}$$

and

$$F(x(t)) < (dh/4)(M/d)^{1/j}. \tag{17}$$

Since $q(t)/r(t) \to 1/M$ as $t \to \infty$, there exists $b > a$ such that

$$q(t)/r(t) < 2/M \tag{18}$$

for $t > b$. Choose $T > b$ so that $x(T) < 0$ and $x'(T) = 0$. If $t > T$ and $x(t) < 0$, then the conclusion of the theorem follows from (16). If $t > T$ and $x(t) \geq 0$, let $V(x, y, t) = y^2/2r(t) + q(t) F(x)/r(t)$, and by differentiating and integrating V we obtain $q(t) F(x(t))/r(t) \leq x(t) - x(T) + + q(T) F(x(T))/r(T)$. Hence, by (11), $d[x(t)]^{1+j} < (M/d)^{1/j}(dh) + Mx(t)$.

Thus

$$x(t)\,[x^j(t) - M/d] < (M/d)^{1/j}\,h\,. \tag{19}$$

If $x^j(t) \leq M/d$, then the theorem follows. If $x^j(t) > M/d$ from (19) we have $x^j(t) - M/d < h$, and the proof of the theorem is now complete.

REMARK 1. If in equation (1), $f(x) = x^p$ where p is an odd positive integer, then we may take $d = 1/(p+1)$ and $j = p$ in condition (11) above. In this case the conclusion of Theorem 4 has the form

$$-h < x(t) < [(p+1)M + h]^{1/p}.$$

REMARK 2. It is interesting to note that the conclusions of Theorems 3 and 4 do not hold if the monotonicity condition of $H(t)$ is relaxed. The equation

$$x'' + 4t^2 x = 8t^2 + 6\cos t^2, \qquad t \geq 1$$

satisfies all the hypotheses of these theorems except that $H(t) = 2 + 3(\cos t^2)/2t^2$ is not monotonic. Here $M = 2$ and we can take $d = 1/2$ and $j = 1$. Now $x(t) = 2 + 3\sin t^2$ is a solution of this equation which does not satisfy $-h < x(t) < 4 + h$ for any $h < 1$.

The following two theorems generalize Theorems 2.1 and 2.2 in [2].

THEOREM 5. *Suppose conditions* (4), (6), (9) and (10) *hold. If* $x(t)$ *is a nonoscillatory solution of* (1), *then either* $x(t)$ *is ultimately monotonic, or there exists* $a > t_0$ *such that* $x(t)$ *has a positive lower bound for* $t \geq a$.

Proof. Let $x(t)$ be a nonoscillatory solution of (1). By Lemma 4.1 in [1], there exists $a > t_0$ such that $x(t) > 0$ for $t \geq a$. Assume that $x(t)$ is not ultimately monotonic and that $x(t)$ does not have a positive lower bound on $[a, \infty)$. Choose $b \geq a$ such that $H(t) = r(t)/q(t) > 3M/4$ for $t \geq b$. By (9) there exists $A > 0$ such that $f(x) < M/4$ for $0 < x < A$. Also, there exists $c > b$ such that $y(c) = 0$ and $x(c) < A$, and there exists $d > c$ such that $y(d) = 0$ and $x(d) < x(c)/4$. Defining $W_c(x, y, t)$ as in the proof of Theorem 3 we have $W_c(d) \leq W_c(c)$ from which we obtain

$$\int_c^d H(s)\,y(s)\,ds > -F(x(c))\,. \tag{20}$$

But, by a mean value theorem for integrals, there is a number w in $[c, d]$ such that

$$\int_c^d H(s)\,y(s)\,ds = x(w)\,[H(c) - H(d)] + H(d)\,x(d) - H(c)\,x(c) < -Mx(c)/2$$

which contradicts (20) since $F(x(c)) \leq f(x(c))\,x(c) < Mx(c)/4$.

REMARK 3. An example showing that the monoticity condition on $H(t)$ in Theorem 5 is essential can be found in [2].

THEOREM 6. *Suppose conditions* (2), (4), (6) *and* (9)–(11) *hold. Let* $x(t)$ *be a solution of* (1). *Then*:

(i) *If* $x(t)$ *is ultimately monotonic, then* $f(x(t)) \to \mathrm{M}$ *as* $t \to \infty$.

(ii) *If* $x(t)$ *is nonoscillatory, then there exists* $h > 0$ *and* $\mathrm{T} > t_0$ *such that* $h \leq x(t) \leq (\mathrm{M}/d)^{1/j}$ *for* $t \geq \mathrm{T}$.

Proof. Assume that $x(t)$ is a solution of (1) which is ultimately monotonic. By Lemma 4.1 in [1] there exists $a > t_0$ such that $x(t) > 0$ for $t \geq a$. By Theorem 1 we have that $x(t)$ is bounded. Since $x(t)$ is monotonic, $f(x(t)) \to \mathrm{A}$ by (10). If $\mathrm{A} \neq \mathrm{M}$, an integration of the second equation in (5) leads to a contradiction.

To prove (ii) we first assume that $x(t)$ is ultimately monotonic. Then there exist $\mathrm{T} > t_0$ and $h > 0$ such that $\mathrm{M}/2 \leq f(x(t)) \leq \mathrm{M}$, $f(h) = \mathrm{M}/2$ and $x(t) \geq h$ for $t \geq \mathrm{T}$. Now

$$d\,[x(t)]^{1+j} \leq \mathrm{F}(x(t)) \leq f(x(t))\,x(t) \leq \mathrm{M}x(t) \tag{21}$$

for $t \geq \mathrm{T}$ so $x(t) \leq (\mathrm{M}/d)^{1/j}$ for $t \geq \mathrm{T}$ and (ii) holds.

If $x(t)$ is not ultimately monotonic, then by Theorem 5 there exists $h > 0$ and $a > t_0$ such that $x(t) > h$ for $t \geq a$. If $f(x(t)) \leq \mathrm{M}$ for $t \geq a$, then, by (21), (ii) holds. Suppose that there exists $b \geq a$ such that $f(x(b)) > \mathrm{M}$. Then there exists $c > b$ such that $f(x(c)) < \mathrm{M}$, for otherwise it can be shown that $x(t)$ must be monotonic. Now if $f(x(t)) \leq \mathrm{M}$ for $t \geq c$, we are done so suppose there exists d $> c$ such that $f(x(d)) > \mathrm{M}$. Notice that $x(c) < x(d)$ since f is monotonic. Since f is continuous there exists K such that $x(c) < \mathrm{K} < x(d)$, $f(\mathrm{K}) = \mathrm{M}$, and $f(x(t)) > \mathrm{M}$ for $\mathrm{K} < x(t) \leq x(d)$. There exists $d_1 > c$ such that $x(d_1) = \mathrm{K}$ and $x(t) > \mathrm{K}$ for $d_1 < t \leq d$ since $x(t)$ is continuous. Since there is again a value of t for which $f(x(t)) < \mathrm{M}$, a similar argument will yield $d_2 > d$ with the property that $x(d_2) = \mathrm{K}$ and for $d \leq t < \mathrm{d}_2$ we have $f(x(t)) > \mathrm{M}$ and $x(t) > \mathrm{K}$. If $x(t)$ attains its maximum value on $[d_1, d_2]$ at d_3, then d_3 is in (d_1, d_2) and $y(d_3) = 0$. Now on (d_1, d_2), $y'(t) < 0$ so $y(d_2) < 0$. Let $\mathrm{T} > d_2$ be such that $y(\mathrm{T}) = 0$ and $y(t) < 0$ for $\mathrm{d}_2 < t < \mathrm{T}$. Since $y(\mathrm{T}) = y(d_3)$ and since $y'(t) < 0$ on (d_3, d_2), we must have $\mathrm{H}(m) > f(x(m))$ for some m in (d_2, T). We have $y(t) < 0$ on (d_2, T) so $x(t)$ is decreasing on this interval and hence $f(x(\mathrm{T})) \leq$ $\leq f(x(m)) < \mathrm{H}(m) \leq \mathrm{H}(\mathrm{T})$. Define $\mathrm{V}(x, y, t) = y^2/2\,q(t) + \mathrm{F}(x)$; then $\mathrm{V}' \leq yr(t)/q(t)$. Integrating we have $\mathrm{V}(t) \leq \mathrm{V}(\mathrm{T}) + \mathrm{H}(t)\,x(t) - \mathrm{H}(\mathrm{T})\,x(\mathrm{T}) +$ $+ x(w)\,[\mathrm{H}(\mathrm{T}) - \mathrm{H}(t)]$ where $\mathrm{T} \leq w \leq t$. Hence $d\,[x(t)]^{1+j} \leq f(x(\mathrm{T}))\,x(\mathrm{T})$ $+ \mathrm{H}(t)\,x(t) - \mathrm{H}(\mathrm{T})\,x(\mathrm{T}) < \mathrm{M}x(t)$ for $t \geq \mathrm{T}$ and so (ii) holds.

The equation $x'' + x = 1$ demonstrates that the bounds in part (ii) of Theorem 6 are sharp.

REFERENCES

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