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On Coddington and Levinson's results for a nonlinear boundary value problem involving a small parameter

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Equazioni differenziali ordinarie. — *On Coddington and Levinson's results for a nonlinear boundary value problem involving a small parameter* (*). Nota di K. W. CHANG, presentata (**) dal Socio G. SANSONE.

RIASSUNTO. — Supposto $f, g \in C^1$, $f \geq \mu > 0$, $f(t, y) y' + g(t, y) = 0$ abbia una soluzione $y = \bar{y}(t)$ tale che $\bar{y}(1) = \beta$, Coddington e Levinson hanno dimostrato che per ε sufficientemente piccolo, il sistema non lineare

$$\varepsilon y'' + f(t, y) y' + g(t, y) = 0 \quad , \quad y(0) = \alpha \quad , \quad y(1) = \beta$$

ha una soluzione $y = y(t, \varepsilon)$ in $[0, 1]$ e inoltre

$$y(t, \varepsilon) \to \bar{y}(t) \quad , \quad y'(t, \varepsilon) \to \bar{y}'(t) \qquad \text{per} \quad 0 < \delta \leq t \leq 1 .$$

Con un nuovo metodo si dimostra che se $f, g \in C^2$ allora

$$y(t, \varepsilon) = \bar{y}(t) + O(\varepsilon) + O(e^{-\mu t/\varepsilon}) \quad , \quad y'(t, \varepsilon) = \bar{y}'(t) + O(\varepsilon) + O(\varepsilon^{-1} e^{-\mu t/\varepsilon}) \qquad \text{per} \quad 0 \leq t \leq 1.$$

1. STATEMENT OF RESULTS

In [1], Coddington and Levinson studied the nonlinear boundary value problem

$$\varepsilon y'' + f(t, y) y' + g(t, y) = 0 \tag{1}$$

$$y(0) = \alpha \quad , \quad y(1) = \beta , \tag{2}$$

where ε is a small positive parameter. It is natural to expect that an approximate solution to the problem (1), (2) will be given by the function $\bar{y}(t)$ which satisfies the degenerate equation

$$f(t, \bar{y}) \bar{y}' + g(t, \bar{y}) = 0$$

and one of the boundary conditions (2). In fact they obtained the following result.

THEOREM 1. *Suppose*

(I) *the functions* $f(t, y)$ *and* $g(t, y)$ *are such that the problem*

$$f(t, \bar{y}) \bar{y}' + g(t, \bar{y}) = 0 \quad , \quad \bar{y}(1) = \beta$$

has a solution $\bar{y}(t)$ *on* $0 \leq t \leq 1$;

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(II) *f, g are of class C^1 in a region*

$$R_\sigma = \{(t, y) : 0 \le t \le 1, |y - \bar{y}(x)| \le \sigma, \sigma > 0\}$$

which includes the point $(0, \alpha)$;

(III) *there exists a constant* $\mu > 0$ *such that*

$$f(t, y) \ge \mu \quad \text{in} \quad R_\sigma .$$

Then, for ε *sufficiently small, a solution* $y(t) = y(t, \varepsilon)$ *of* (1), (2) *exists in* R_σ *such that* $y(t, \varepsilon) \to \bar{y}(t)$ *and* $y'(t, \varepsilon) \to \bar{y}'(t)$ *uniformly on any subinterval* $0 < \delta \le t \le 1$. *Moreover, in a region* R_{σ_0} $(0 < \sigma_0 < \sigma)$, *there exists at most one solution of* (1), (2).

The behaviour of the solution in the boundary layer (i.e. near the end-point $t = 0$) was not discussed in [1]. Wasow [5] and Erdélyi [2] studied this boundary layer behaviour and showed that there exists a solution $y(t, \varepsilon)$ such that

$$(3) \qquad \begin{aligned} y(t, \varepsilon) &= \bar{y}(t) + O(\varepsilon) + O(e^{-\mu t/\varepsilon}) \\ y'(t, \varepsilon) &= \bar{y}'(t) + O(\varepsilon) + O(\varepsilon^{-1} e^{-\mu t/\varepsilon}) \end{aligned}$$

uniformly throughout the whole interval $0 \le t \le 1$. However, they needed an extra assumption, namely, some restriction on the size of the boundary layer jump $|\alpha - \bar{y}(0)|$. Recently, by assuming f, g to have power series expansions in $\tau = t/\varepsilon$, O'Malley [4] removed this restriction on the smallness of $|\alpha - \bar{y}(0)|$.

In this paper we show that the results of Theorem 1 can be extended to include the results (3), without requiring $|\alpha - \bar{y}(0)|$ to be small or f, g to be analytic. We require only that f, g are twice continuously differentiable.

THEOREM 2. *Suppose assumptions* (I) *and* (III) *of Theorem 1 hold and suppose* (II') *f and g are of class C^2 in R_σ. Then for ε sufficiently small, a solution $y(t, \varepsilon)$ of* (1), (2) *exists in R_σ and*

$$(3) \qquad \begin{aligned} y(t, \varepsilon) &= \bar{y}(t) + O(\varepsilon) + O(e^{-\mu t/\varepsilon}) \\ y(t, \varepsilon) &= \bar{y}'(t) + O(\varepsilon) + O(\varepsilon^{-1} e^{-\mu t/\varepsilon}) \end{aligned}$$

uniformly for the interval $0 \le t \le 1$. *Moreover, in a region* R_{σ_0} *(with* $0 < \sigma_0 < \sigma$), *there exists at most one solution of* (1), (2).

In view of Theorem 1, it suffices to prove only the results (3) of Theorem 2. Our method of proof is to replace the problem (1), (2) by a more tractable problem for a diagonalized system of two first order equations. This approach obviates the necessity, encountered by earlier writers, of breaking up the error term into the so-called inner and outer corrections.

We note that the boundary value problem

$$\varepsilon y'' + y' + y^{n+1} = 0 \quad , \quad y(0) = \alpha \quad , \quad y(1) = \beta$$

was cited by Willet [6] and Erdélyi [3] to illustrate the usefulness of employing a more refined approximate solution. It follows from Theorem 2 that for the whole class of problems (1), (2) (of which the above problem is a special case) the existence of $\bar{y}(t)$ already ensures results (3); therefore it is not necessary to employ approximate solutions more refined than $\bar{y}(t)$.

2. Transformation into a Diagonalized System

Set $v = y(t, \varepsilon) - \bar{y}(t)$, where $y(t, \varepsilon)$ is the solution of Theorem 1. Then by equation (1) we have

$$\varepsilon v'' = -f(t, y(t, \varepsilon))(\bar{y}' + v') - g(t, \bar{y}(t) + v) - \varepsilon \bar{y}'',$$

or equivalently,

$$\varepsilon v'' + A(t)\, v' + B(t)\, v = \tilde{G}(t, \varepsilon, v), \tag{4}$$

where

$$A(t) = A(t, \varepsilon) = f(t, y(t, \varepsilon)),$$
$$B(t) = f_y(t, \bar{y}(t))\, \bar{y}' + g_y(t, \bar{y}(t)),$$

and

$$\tilde{G}(t, \varepsilon, v) = -f(t, \bar{y}(t) + v)\, \bar{y}' - g(t, \bar{y}(t) + v) + B(t)\, v - \varepsilon \bar{y}''.$$

Let $p(t) = p(t, \varepsilon)$ be the solution of

$$\varepsilon p' = -A(t)\, p - \varepsilon p^2 - B(t), \quad p(0) = 0, \tag{5}$$

and let $q(t) = q(t, \varepsilon)$ be the solution of

$$\varepsilon q' = [A(t) + 2\varepsilon p(t)]\, q - 1, \quad q(1) = 0. \tag{6}$$

At the end of this section it will be shown that, for ε sufficiently small the solutions $p(t), q(t)$ exist and are bounded on $[0, 1]$. With these functions $p(t), q(t)$ let us introduce the new variables

$$z = v' - p(t)\, v, \tag{7 a}$$

$$w = v + \varepsilon q(t)\, z. \tag{7 b}$$

Applying the change of variable (7 a) to the equation (4) we obtain the following "triangular" system

$$v' = p(t)\, v + z$$
$$\varepsilon z' = -[A(t) + \varepsilon p(t)]\, z + \tilde{G}(t, \varepsilon, v)$$

which reduces, under the further change of variable (7 b), to the "separated" or diagonalized system

$$\begin{aligned} w' &= p(t)\, w + q(t)\, G(t, \varepsilon, w, z), \\ \varepsilon z' &= -[A(t) + \varepsilon p(t)]\, z + G(t, \varepsilon, w, z), \end{aligned} \tag{8}$$

where

$$G(t, \varepsilon, w, z) = \tilde{G}(t, \varepsilon, w - \varepsilon q(t) z).$$

Similarly, under (7 b) the boundary conditions (2) become

$$(9) \qquad \begin{aligned} w(0) - \varepsilon q(0) z(0) &= v(0) = \alpha - \bar{y}(0), \\ w(1) = w(1) - \varepsilon q(1) z(1) &= v(1) = 0, \end{aligned}$$

because $q(1) = 0$.

The diagonalized system (8) is more tractable than the equation (4); in fact, the general solution of (8) satisfies the integral equations

$$w(t) = P(t) C_1 - \int_t^1 P(t) P^{-1}(s) q(s) G[s, \varepsilon, w(s), z(s)] ds$$

$$z(t) = Z(t) C_2 + \varepsilon^{-1} \int_0^t Z(t) Z^{-1}(s) P^{-1}(t) P(s) G[s, \varepsilon, w(s), z(s)] ds$$

where

$$P(t) = \exp\left(\int_0^t p(s) ds\right),$$

$$Z(t) = Z(t, \varepsilon) = \exp\left\{-\varepsilon^{-1} \int_0^t f[s, y(s, \varepsilon)] ds\right\}$$

and C_1, C_2 are arbitrary constants. It only remains to choose C_1 and C_2 so as to satisfy the boundary conditions (9). The end results will appear as equations (12) in the next section.

The existence of bounded nonzero functions $p(t), q(t)$ follow from the

LEMMA. *There exists $\varepsilon_1 > 0$ such that the equations* (5) *and* (6) *have, respectively, solutions $p(t)$ and $q(t)$ which are uniformly bounded for $0 \leqq t \leqq 1$, $0 < \varepsilon \leqq \varepsilon_1$.*

Proof. For any continuous function $F(t)$ defined on $[0, 1]$ we set

$$\| F(t) \| = \max_t | F(t) |.$$

It can be verified by differentiation that a solution $p(t)$ of the integral equation $p(t) = Tp(t)$ where

$$Tp(t) = \int_0^t Z(t) Z^{-1}(s) [-p^2(s) - \varepsilon^{-1} B(s)] ds$$

is also a solution of the differential equation in (5). Moreover $p(0) = 0$. We will prove the existence of a unique solution of the integral equation by the contraction mapping principle. In fact, by (III)

$$|\mathrm{T}p(t) - \mathrm{T}p^*(t)| \leqq \|p(t) + p^*(t)\| \, \|p(t) - p^*(t)\| \int_0^t e^{-\mu(t-s)/\varepsilon} \, ds$$

and hence

$$\|\mathrm{T}p(t) - \mathrm{T}p^*(t)\| \leqq \varepsilon\mu^{-1} \|p(t) + p^*(t)\| \, \|p(t) - p^*(t)\| .$$

Choose $\varepsilon_1 > 0$ so small that

$$\varepsilon_1 (2\mu^{-1})^2 \|\mathrm{B}(t)\| \leqq 1/2.$$

If $\|p\|, \|p^*\| \leqq \rho$, where $\rho = 2\mu^{-1}\|\mathrm{B}(t)\|$, it follows immediately that for $0 < \varepsilon \leqq \varepsilon_1$

$$\|\mathrm{T}p(t) - \mathrm{T}p^*(t)\| \leqq 2\varepsilon\mu^{-1}\rho \|p(t) - p^*(t)\| \leqq 1/2 \, \|p(t) - p^*(t)\| .$$

Similarly we obtain

$$\|\mathrm{T}p(t)\| \leqq \mu^{-1}\varepsilon\rho^2 + \mu^{-1}\|\mathrm{B}(t)\| \leqq \rho/4 + \rho/2 < \rho .$$

Therefore it follows by the contraction principle that the integral equation $p(t) = \mathrm{T}p(t)$ has a unique solution $p(t)$ in $\|p(t)\| \leq \rho$.

We next obtain a bounded nontrivial solution $q(t)$ of the equation (6). In fact, by integrating (6) between the limits t and 1 the required solution is

$$q(t) = \varepsilon^{-1} \int_t^1 \exp\left(-\varepsilon^{-1} \int_t^s [\mathrm{A}(\tau) + 2\varepsilon p(\tau)] \, d\tau\right) ds .$$

Since ε_1 has been chosen so small that $\varepsilon_1 \rho \leq \mu/4$, it follows that for $0 \leqq t \leqq 1$ and $0 < \varepsilon \leqq \varepsilon_1$

$$q(t) \leqq (\mu - 2\varepsilon\rho)^{-1} \leqq 2/\mu .$$

Note that if we set $t = 0$ in the integral for $q(t)$ and integrate by parts, then $q(0) > 0$.

3. Proof of Theorem 2

As we have mentioned, it suffices to prove only the results (3) of Theorem 2. If $\|\mathrm{P}(t)\| \leq \rho$ as in the proof of the Lemma, it follows that the function

$$\mathrm{P}(t) = \exp\left\{\int_0^t p(s) \, ds\right\}$$

which is the solution of $x' = p(t)x$ such that $\mathrm{P}(0) = 1$, satisfies

(10) $$|\mathrm{P}(t)\,\mathrm{P}^{-1}(s)| \leq e^{\rho} = \mathrm{L} \quad \text{for} \quad 0 \leq t, s \leq 1$$

By assumption (III), we have

(11) $$\mathrm{Z}(t)\,\mathrm{Z}^{-1}(s) \leq \exp(-\mu(t-s)/\varepsilon) \quad \text{for} \quad 0 \leq s \leq t \leq 1 .$$

By assumption (II′) and the Lemma, there exists a positive constant k such that

$$(|f_{yy}|\,|\bar{y}'|) + |g_{yy}|)(1 + \|q(t)\|)^2 \leq k/4$$

and we obtain, by applying the mean value theorem twice as in [2],

$$|\,G(t, \varepsilon, w_1, z_1) - G(t, \varepsilon, w_2, z_2)\,| \leq kh(w_1, w_2, z_1, z_2)$$

where

$$h(w_1, w_2, z_1, z_2) = \max(|w_1|, |w_2|, \varepsilon|z_1|, \varepsilon|z_2|)\max(|w_1 - w_2|, \varepsilon|z_1 - z_2|).$$

Also, there exists $c > 0$ such that

$$|\,G(t, \varepsilon, 0, 0)\,| = \varepsilon\,|\bar{y}''| \leq c\varepsilon .$$

It can readily be verified that the solution of the problem (8), (9) satisfies the integral equations

$$w(t) = -\int_t^1 P(t)\,P^{-1}(s)\,q(s)\,G[s, \varepsilon, w(s), z(s)]\,ds \tag{12}$$

$$z(t) = \varepsilon^{-1} Z(t)\,P^{-1}(t)\,[w(0) - \alpha + \bar{y}(0)]/q(0)$$

$$+ \varepsilon^{-1}\int_0^t Z(t)\,Z^{-1}(s)\,P^{-1}(t)\,P(s)\,G[s, \varepsilon, w(s), z(s)]\,ds .$$

We shall prove the existence of a solution of (12) with the required properties by the method of successive approximations, setting $(w_0, z_0) = (0, 0)$, $(w_n, z_n) = T(w_{n-1}, z_{n-1})$, $n = 1, 2, \cdots$ where

$$Tw(t) = -\int_t^1 P(t)\,P^{-1}(s)\,q(s)\,G[s, \varepsilon, w(s), z(s)]\,ds \tag{13}$$

$$Tz(t) = \varepsilon^{-1} Z(t)\,P^{-1}(t)\,[Tw(0) - \alpha + \bar{y}(0)]/q(0)$$

$$+ \varepsilon^{-1}\int_0^t Z(t)\,Z^{-1}(s)\,P^{-1}(t)\,P(s)\,G[s, \varepsilon, w(s), z(s)]\,ds .$$

We collect here the following elementary estimates while will be used:

$$\int_0^t Z(t)\,Z^{-1}(s)\,ds \leq \varepsilon/\mu ,$$

$$\int_0^t Z(t)\,Z(s)\,ds \leq \varepsilon Z(t)/\mu , \tag{14}$$

$$\int_t^1 Z^m(s)\,ds \leq \varepsilon Z^m(t)/m\,\mu , \qquad \text{if} \quad m > 0 .$$

Take

(15) $$c_0 = \sup_{t,\varepsilon} [c \mid q \mid , (\mathrm{L}c \mid q \mid + \mid \alpha - \bar{y}(0) \mid)/q(0)] ,$$
$$c_1 = \mathrm{L}(c_0 + \mu^{-1})$$

and

(16) $$\lambda = 4\, c_1 \mathrm{L}k(1 + 2\,\mu^{-1})(1 + \mathrm{L} \| q \| / q(0)) .$$

Finally choose ε_0 so small that

$$\lambda\varepsilon_0 \le 1/2 \qquad \text{for} \quad 0 < \varepsilon \le \varepsilon_0 .$$

In view of (10), (14) and (15) we obtain

$$| w_1(t) | \le \mathrm{L}c \| q \| \varepsilon \le c_1 (\mathrm{Z}(t) + \varepsilon)$$

$$\varepsilon \,| z_1(t) | \le \mathrm{L} \mid \alpha - \bar{y}(0) \mid c\,\mathrm{Z}(t)/q(0) + \mathrm{L}c\mu^{-1}\varepsilon^2 \le c_1(\mathrm{Z}(t) + \varepsilon^2) .$$

Also, from (13) we have

$$| w_{a+1}(t) - w_n(t) | \le \mathrm{L}k \| q \| \int_t^1 h(w_n(s), w_{n-1}(s), z_n(s), z_{n-1}(s))\, ds$$

$$\varepsilon \,| z_{n+1}(t) - z_n(t) | \le \mathrm{Z}(t)\, \mathrm{L} \mid w_{n+1}(0) - w_n(0) \mid / q(0)$$
$$+ \mathrm{L}k \int_0^t \mathrm{Z}(t)\,\mathrm{Z}^{-1}(s)\, h(w_n(s), w_{n-1}(s), z_n(s), z_{n-1}(s))\, ds$$

and it follows by induction that

$$| w_n(t) - w_{n-1}(t) | \le (\lambda\varepsilon)^{n-1} c_1(\mathrm{Z}(t) + \varepsilon) ,$$
$$\varepsilon \,| z_n(t) - z_{n-1}(t) | \le (\lambda\varepsilon)^{n-1} c_1(\mathrm{Z}(t) + \varepsilon^2) ,$$
$$| w_n(t) | \le 2\, c_1(\mathrm{Z}(t) + \varepsilon) ,$$
$$\varepsilon \,| z_n(t) | \le 2\, c_1(\mathrm{Z}(t) + \varepsilon^2) ,$$

where λ is defined by (16). Therefore, since $\lambda\varepsilon \le 1/2$, the series

$$\sum_{n=1}^{\infty} [w_n(t) - w_{n-1}(t)] \quad , \quad \sum_{n=1}^{\infty} [z_n(t) - z_{n-1}(t)]$$

converge uniformly in (t, ε) to a solution $w(t), z(t)$ of (12) and for $0 \le t \le 1$

$$| w(t) | \le \sum_{n=1}^{\infty} | w_n(t) - w_{n-1}(t) | \le 2\, c_1(\mathrm{Z}(t) + \varepsilon)$$

$$| z(t) | \le \sum_{n=1}^{\infty} | z_n(t) - z_{n-1}(t) | \le 2\, c_1(\varepsilon^{-1}\mathrm{Z}(t) + \varepsilon) .$$

Returning to the original variable we have for $0 \leq t \leq 1$

$$|y(t, \varepsilon) - \bar{y}(t)| \leq |w(t)| + \varepsilon \|q\| |z(t)| \leq C(Z(t) + \varepsilon) \cdot$$

$$|y'(t, \varepsilon) - \bar{y}'(t)| \leq \|p\| (|w(t)| + \varepsilon \|q\| |z(t)|) + |z(t)|$$

$$\leq C(\varepsilon^{-1} Z(t) + \varepsilon)$$

for some positive constant C, which implies (3).

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