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A Generalization of Bendixson's Negative Criterion

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Equazioni differenziali. — *A Generalization of Bendixson's Negative Criterion.* Nota di RUDOLF KURTH, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Sia G un dominio di $E^n, f: G \to E^n$, $n \geq 2$, $(\text{div} f)(a) \neq 0$ per $x \in G$, e sia P l'insieme di tutti i punti a di G per i quali la soluzione $\Phi(a, t)$ del sistema differenziale $dx/dt = f(x)$, $x = a$ se $t = 0$ è periodica e inoltre $(\Phi(a, t)/t \in E^1)$ è un sottoinsieme di G. In queste ipotesi l'insieme P è misurabile secondo Lebesgue e $\mu(P) = 0$.

Bendixon's Negative Criterion reads: Let G be a domain in the euclidean plane E^2, and $f: G \to E^2$ be a vector function which has continuous partial derivatives of the first order in G such that, for any point x of G,

$$(\text{div} f)(x) \neq 0 .$$

Then the differential equation

$$dx/dt = f(x) ,$$

where t denotes a real variable, has no proper (i.e. non-constant) periodic solution whose solution path (i.e. its range) lies entirely in G. (See, e.g., [2]).

For the proof the hypothesis that G is 2-dimensional appears indispensable. But a somewhat weaker version of the theorem can be generalized to euclidean spaces of more than two dimensions in the following fashion:

(1) *Let* G *be a domain in the n–dimensional euclidean space* E^n, *and* $f: G \to E^n$ *be a vector function which has continuous partial derivatives of the first order in* G *such that, for any point x of* G,

$$(\text{div} f)(x) > 0 .$$

Further, let P *be the set of all those points a of* G *for which the solution* $\Phi(a, t)$ *of the initial-value problem*

$$\begin{cases} dx/dt = f(x) , \\ x = a \quad \text{if } t = 0 \end{cases}$$

is periodic and the corresponding solution path $\{\Phi(a, t) \mid t \in E^1\}$ *is a subset of the domain* G. *Then* P *is a Lebesgue-measurable set of measure* $\mu P = 0$

The assumption that div$f$ is positive does not restrict the generality of the assertion. Note that the set P also contains the starting points of the constant solutions, whereas in Bendixson's theorem these points are excluded.

Proof. There is an increasing sequence $\{C_i\}_{i=1}^{\infty}$ of compact subsets $C_1, C_2, \cdots$ of G such that

$$G = \bigcup_{i=1}^{\infty} C_i$$

(*) Nella seduta del 14 aprile 1973.

and, for any $i = 1, 2, \cdots$, the boundaries of C_i and G have a distance $\geq 1/i$. Let P_i be the set of all those starting points of periodic solutions whose paths belong entirely to C_i. Then,

$$P \supseteq \bigcup_{i=1}^{\infty} P_i .$$

An indirect argument, using the compactness of the solution paths, shows that even

$$P = \bigcup_{i=1}^{\infty} P_i .$$

Hence it suffices to prove that, for $i = 1, 2, \cdots$,

$$\mu P_i = 0 .$$

The closure $\overline{P}_i$ of P_i is a measurable set, and so is $\Phi(\overline{P}_i, t)$ for any $t \in E^1$. By Liouville's Theorem,

$$\mu\Phi(\overline{P}_i, t) = \int_{\overline{P}_i} \exp \left\{ \int_0^t (\operatorname{div} f)(\Phi(a, \tau)) \, d\tau \right\} da .$$

(See, e.g., [1]). Let

$$\alpha_i = \operatorname{Min} \{ (\operatorname{div} f)(x) \mid x \in C_i \} .$$

By the hypotheses made, $\alpha_i > 0$. It follows that

$$\mu\Phi(\overline{P}_i, t) \geq e^{\alpha_i t} \cdot \mu\overline{P}_i .$$

On the other hand, since $a \in P_i$ implies that $\Phi(a, t) \in P_i$ for all t, the set P_i is invariant under the flow Φ, and so is its closure $\overline{P}_i$:

$$\Phi(\overline{P}_i, t) = \overline{P}_i \qquad \text{for all } t.$$

Hence, by the preceding inequality,

$$\mu\overline{P}_i \geq e^{\alpha_i t} \cdot \mu\overline{P}_i \qquad \text{for all } t.$$

With P_i also $\overline{P}_i$ is a subset of the bounded measurable set C_i; therefore, $\mu\overline{P}_i < \infty$. It follows, because $\alpha_i > 0$, that

$$\mu\overline{P}_i = 0 .$$

P_i, as a subset of a null-set, is measurable, and

$$\mu P_i = 0 ,$$

as has been asserted.

The decisive point of the proof is that, for any point a of P, the solution path (or, rather, its closure) is a compact subset of the domain G. Hence the following generalization of the above proposition holds under the same hypotheses:

(2) *Let* Q^+ *be the set of all those points* b *of* G *for which the solution* $\Phi(b, t)$ *is defined for all positive real* t *and the closure of the solution path* $\{\Phi(b, t) \mid t \geq 0\}$ *is a compact subset of* G. *Then* Q^+ *is a null-set.*

There is a similar proposition holding for negative real t if $\operatorname{div} f$ is negative in G.

A point c of G is said to be a recurrence point of the flow with respect to the future if

$$c \in \overline{\{\Phi(c, k) \mid k = 1, 2, 3, \cdots\}}.$$

The following specialization of (2) is still a generalization of (1) and, therefore, of Bendixson's Negative Criterion:

(3) *Let* R^+ *be the set of all those recurrence points* c *with respect to the future for which* $\{\Phi(c, t) \mid t \geq 0\}$ *is a compact subset of* G. *Then* R^+ *is measurable and has measure* 0.

Again, there is a similar proposition for the past, i.e. $t < 0$.

REFERENCES

[1] KURTH R., *Axiomatics of Classical Statistical Mechanics*. Pergamon Press, 1960, p. 55.
[2] WILSON H. K., *Ordinary Differential Equations*. Addison-Wesley, 1971, p. 276.