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Derivation of Modified Thomas-Fermi and Emden Equations

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Equazioni differenziali. — *Derivation of Modified Thomas-Fermi and Emden Equations.* Nota di JAMES L. REID e RICHARD J. DE PUY, presentata (*) dal Socio M. PICONE.

RIASSUNTO. — Si scrive l'equazione differenziale non-lineare del 2° ordine soddisfatta dalla funzione omogenea $y = [\pm m(au^{\beta} v^{\gamma} + bu^{j} v^{n})]^{k/m}$. Le funzioni u e v sono soluzioni dell'equazione lineare $y'' + r(x) y' + q(x) y = 0$; a e b sono costanti arbitrarie; e gli esponenti sono reali e non nulli. Come casi particolari delle equazioni ottenute in tal modo, si ha un'equazione modificata di Thomas–Fermi $y'' = (1 + c_1 x^{\alpha} + c_2 x^{2\alpha}) x^{-1/2} y^{3/2}$, ed un'equazione modificata di Emden: $y'' = (1 + C_1 x^{\alpha} + C_2 x^{2\alpha}) x^{1-M} y^{M}$. Le costanti c_1, c_2 e C_1, C_2 sono date esplicitamente.

Recently the first author has shown [1] that the function $y = [\pm bmu^{j} v^{n}]^{k/m}$, $m = j + n$, which is homogeneous of degree k, satisfies the nonlinear differential equation

$$y'' + r(x) y' + kq(x) y = (1 - l) y'^{2} y^{-1} - b^{2} knju^{2j-2} v^{2n-2} W^{2} y^{1-2ml}, \tag{1}$$

$kl = 1$, provided that the functions $u(x)$ and $v(x)$ are independent solutions of the linear differential equation

$$y'' + r(x) y' + q(x) y = 0 . \tag{2}$$

The function $W(x) \neq 0$ is the Wronskian of u and v, and the exponents are real and non-zero. It was shown that the Thomas-Fermi equation $y'' = y^{3/2}/x^{1/2}$ follows from (1) without appeal to physical arguments.

The object of the present Note is to obtain the explicit form of the nonlinear differential equation when the homogeneous function

$$y = [\pm m(au^{\beta} v^{\gamma} + bu^{j} v^{n})]^{k/m}, \qquad m = j + n = \beta + \gamma , \tag{3}$$

is assumed to satisfy the nonlinear differential equation

$$L_k y(x) = R(x, y(x), y'(x)) , \qquad kl = 1 , \tag{4}$$

where

$$L_k = d^2/dx^2 + r(x) d/dx + kq(x) \tag{5}$$

is a linear operator and $R(x, y, y')$ is to be determined. Again we require that u and v satisfy the linear equation

$$L_1 Y = 0 , \tag{6}$$

where L_1 is (5) with $k = 1$. We find the explicit form of (4), and we show that a class of nonlinear equations related to the Thomas-Fermi-Emden equations follows immediately.

(*) Nella seduta del 14 aprile 1973.

We start the derivation by substituting the function (3) into the equation (4). Following [1], we define

$$(7) \qquad A \equiv a(\beta u^{\beta-1} u' v^{\gamma} + \gamma u^{\beta} v^{\gamma-1} v') + b(j u^{j-1} u' v^{n} + n u^{j} v^{n-1} v')$$

and express y' and y'' in terms of A and A′. Thus (4) becomes

$$(8) \qquad k(l-m) A^2 + k y^{ml} [A' + r(x) A + q(x) y^{ml}] = y^{2ml-1} R .$$

The details of reducing (8) are quite similar to those of [1], though somewhat more tedious. Making use of (6) and of the expression

$$(9) \qquad A = y^{ml-1} y',$$

we obtain the nonlinear differential equation

$$(10) \qquad L_k y = (1-l) y'^2 y^{-1} - k \{ a^2 \beta\gamma u^{2\beta-2} v^{2\gamma-2} + b^2 jn u^{2j-2} v^{2n-2} + ab [j\gamma + n\beta - m(n-\gamma)^2] u^{j+\beta-2} v^{n+\gamma-2} \} W^2 y^{1-2ml},$$

satisfied by (3). We point out that $(n-\gamma)^2 = (j-\beta)^2$. It is obvious that if either of the arbitrary constants a or b is zero then (10) reduces to the result in [1]. We note also that if $\gamma = n$, then $\beta = j$, and (10) reduces to the result in [1] with b replaced there by $a + b$. If, on the other hand, $\beta = n$ and $\gamma = j$, we obtain

$$(11) \qquad L_k y = (1-l) y'^2 y^{-1} - k \{ jn [a^2 u^{2n-2} v^{2j-2} + b^2 u^{2j-2} v^{2n-2}] + ab [j^2 + n^2 - m(n-j)^2] u^{m-2} v^{m-2} \} W^2 y^{1-2ml}.$$

When $m = 2$ and $l = 1$ in (11) we have the Pinney equation [2]

$$(12) \qquad L_1 y = - \{ jn [a^2 u^{2n-2} v^{2j-2} + b^2 u^{2j-2} v^{2n-2}] + ab [(j-n)^2 + 2jn] \} W^2 y^{-3} ;$$

and if $m = 2$ and $j = n = 1$ in (11) we arrive at the equation of J. M. Thomas [3], with $a + b$ replacing b.

We now consider the simple form of L_k when $r(x)$ and $q(x)$ are both identically zero. The functions u and v are thus solutions of the simple differential equation $Y'' = 0$, and we take

$$(13) \qquad u = x \quad , \quad v = 1 \quad , \quad W = uv' - vu' = 1 .$$

For this case, with $a = b$, we may put (10) in the form

$$(14) \qquad y'' = (1-l) y'^2 y^{-1} + b^2 \{ 1 + (jn)^{-1} [j\gamma + n\beta - m(n-\gamma)^2] x^{\beta-j} + \beta\gamma (jn)^{-1} x^{2(\beta-j)} \} x^{2j-2} y^{1-2ml}, \qquad jn \neq 0 ,$$

a particular solution being

$$(15) \qquad y = \left[\frac{mb(1 + x^{\beta-j})}{\pm [kj(j-m)]^{1/2}} \right]^{k/m} x^{jk/m} , \qquad j \neq 0, m .$$

Let α be a real number and suppose that

(16) $$\beta = \alpha + j \quad , \quad \gamma = m - \alpha - j \, ;$$

now (14) becomes

(17) $$y'' = (1 - l)\, y'^2 y^{-1} + b^2 \left[1 + C_1(m, j, \alpha)\, x^{\alpha} + C_2(m, j, \alpha)\, x^{2\alpha}\right] x^{2j-2} y^{1-2ml},$$

where

(18 a) $$C_1(m, j, \alpha) = 2 + [(m - 2j)\,\alpha - m\alpha^2]/j\,(m - j)$$

(18 b) $$C_2(m, j, \alpha) = 1 + [(m - 2j)\,\alpha - m\alpha^2]/j\,(m - j)\,.$$

A particular solution is (15) with $\beta - j$ replaced by α.

Equation (17) leads to a modified Thomas-Fermi equation of the form

(19) $$y'' = (1 + C_1 x^{\alpha} + C_2 x^{2\alpha})\, x^{-1/2} y^{3/2},$$

a solution being

(20) $$y = \frac{144}{x^3 (1 + x^{\alpha})^4}$$

when the following parameters are used with (16):

(21) $$l = 1 \quad , \quad m = -1/4 \quad , \quad j = 3/4 \, , \quad n = -1 \quad , \quad b = 1\,.$$

The coefficients in (19) are

(22) $$C_1 = -(\alpha^2 - 7\alpha - 6)/3 \quad , \quad C_2 = (4\alpha^2 + 7\alpha + 3)/3\,.$$

Comparing (17) with the Emden equation [1, 4, 5]

(23) $$y'' = \pm\, x^{1-M} y^{M}\,,$$

we see that

(24) $$l = 1 \quad , \quad j = (3 - M)/3 \quad , \quad m = (1 - M)/2\,.$$

Hence a modified Emden equation may be written as

(25) $$y'' = \pm \left[1 + C_1(M, \alpha)\, x^{\alpha} + C_2(M, \alpha)\, x^{2\alpha}\right] x^{1-M} y^{M},$$

and a particular solution as

(26) $$y = \left[\frac{(1 - M)(1 + x^{\alpha})}{\pm\,[2\,(3 - M)]^{1/2}}\right]^{\frac{2}{1-M}} x^{(3-M)/(1-M)}\,.$$

The coefficients in this case are

(27 a) $$C_1(M, \alpha) = 2 + [(M - 5)\,\alpha + (M - 1)\,\alpha^2]/(M - 3)\,,$$

(27 b) $$C_2(M, \alpha) = 1 + [(M - 5)\,\alpha + (M - 1)\,\alpha^2]/(M - 3)\,.$$

In conclusion, we observe that Scorza-Dragoni's theorem [6] for the existence and uniqueness of a solution of (14) holds, provided

(28) $$l = 1 \quad , \quad m < 0 \quad , \quad \beta > 1/2 \quad , \quad j > 1/2 - \beta\,.$$

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