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Analisi matematica. — *On a class of nonlinear integro-differential equations.* Nota I di KIN VINH LEUNG, DEMETRIO MANGERON (*), MEHMET NAMIK OĞUZTÖRELI (**) e ROBERTO B. STEIN, presentata (***) dal Socio M. PICONE.

RIASSUNTO. — Gli Autori studiano nel presente lavoro una nuova classe di equazioni integro-differenziali non lineari che costituiscono un modello matematico concernente certe attività elettriche delle reti neuronali.

I. In the present Note we study certain properties of the solutions of a system of nonlinear integro-differential equations of the form

$$(\text{I.1})\quad \begin{cases} \dfrac{dx_i}{dt} = a\left[\dfrac{1}{1+\exp\left\{-f_i t - \sum\limits_{j=1}^{N} c_{ij} x_j\right\}} - x_i\right] + \sum\limits_{k=1}^{K} b_{ik} \int\limits_0^t g_{ik}[x_i(s)]\, e^{-p_{ik}(t-s)}\, ds \\ x_i(0) = x_i^0 \qquad\qquad (i = 1, 2, \cdots, N) \end{cases}$$

where a, b_{ik}, c_{ik} and p_{ik} are certain given constants such that $a > 0$, $c_{ii} = 0$ and $p_{ik} > 0$, $f_i(t)$ and $g_{ik}(x_i)$ are certain given functions which are sufficiently smooth in their arguments, x_i^0's are certain given constants and $x_i = x_i(t)$'s are the unknown functions.

In [1] and [2] the differential system (I.1) has been analysed in the case in which all b_{ik}'s are zero and $1 \leq N \leq \infty$. In the present Note we assume that N is finite, at least one of the b_{ik}'s is different from zero, and that the corresponding $g_{ik}(x_i)$ is not identically equal zero.

Let us note that in certain realistic problems occurring in the study of electrical activities in neural networks we have $a \cong 100$ and $g_{ik}(x_i) \equiv x_i$. The cases $N = K = 1$, and $N = 1$, $K = 2$ are of great importance in the study of isolated neurons for which a very large number of experimental results are available in the literature. The biological implications of our results given here will be presented somewhere else.

In the next sections we shall investigate the existence, uniqueness and the stability of the solutions of the system (I.I) and elaborate a numerical scheme for the construction of the solutions.

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II. To investigate the existence, uniqueness and the stability of the solutions of Eqs. (I.1), we put

$$\text{(II.1)} \qquad \begin{cases} x_{i0} = x_i \\ x_{ik} = \displaystyle\int_0^t g_{ik}[x_i(s)]\, e^{-p_{ik}(t-s)}\, ds \end{cases} \qquad (i = 1, \cdots, N\,;\ k = 1, \cdots, K)\,.$$

Then we can easily show that the integro-differential system (I.1) is equivalent to the following nonlinear ordinary differential system

$$\text{(II.2)} \qquad \begin{cases} \dfrac{dx_{i0}}{dt} = \left[\dfrac{1}{1 + \exp\left\{ -f_i - \sum\limits_{j=1}^{N} c_{ij}\, x_{j0} \right\}} - x_{i0} \right] + \sum\limits_{k=1}^{K} b_{ik}\, x_{ik} \\ \dfrac{dx_{ik}}{dt} = g_{ik}[x_{i0}] - p_{ik}\, x_{ik} \end{cases}$$

and

$$\text{(II.3)} \qquad x_{ik}(0) = \begin{cases} x_i^0 & \text{if } k = 0\,, \\ 0 & \text{if } k \geq 1\,. \end{cases}$$

Clearly, the system (II.2) admits a unique solution $\{x_{ik}(t)\}$ satisfying the initial conditions (II.3), since the functions $g_{ik}(x_{i0})$ and $f_i = f_i(t)$ are supposed to be sufficiently smooth, and N is finite.

Let us note that the system (II.2) is autonomous if $f_i(t) \equiv f_i =$ constant, $i = 1, \cdots, N$. In the next section we investigate this autonomous case in some details.

III. The steady-state solutions of Eqs. (II.2) with $f_i(t) \equiv f_i =$ const. are determined by the following N (K + 1) equations:

$$\text{(III.1)} \qquad \begin{cases} a \left[\dfrac{1}{1 + \exp\left\{ -f_i - \sum\limits_{j=1}^{N} c_{ij}\, x_{j0} \right\}} - x_{i0} \right] + \sum\limits_{k=1}^{K} b_{ik}\, x_{ik} = 0 \\ g_{ik}(x_{i0}) - p_{ik}\, x_{ik} = 0\,, \qquad i = 1, \cdots, N\,;\ k = 1, \cdots, K\,. \end{cases}$$

It can be shown, by using the principle of contraction mappings, that under certain conditions that satisfied by the constants a, b_{ik}, c_{ij}, f_i, $|g'_{ik}[x_{i0}]|$ and p_{ik}, the system (III.1) admits a unique solution $\{x^*_{ik}\}$.

Let $\{x^*_{ik}\}$ be a system of solutions of Eqs. (III.1). We now put

$$\text{(III.2)} \qquad x_{ik} = x^*_{ik} + X_{ik}\,.$$

Then, we find that

$$\text{(III.3)}\qquad \begin{cases} \dfrac{dX_{i0}}{dt} = -aX_{i0} + a\sum_{j=1}^{N} h_{ij} X_{j0} + \sum_{k=1}^{K} b_{ik} X_{ik} + Q_i(X) \\ \dfrac{dX_{ik}}{dt} = g'_{ik}(x^*_{i0}) X_{i0} - p_{ik} X_{ik} + R_{ik}(X) \end{cases}$$

$$(i = 1, \cdots, N\ ;\ k = 1, \cdots, K)$$

where

$$\text{(III.4)}\qquad h_{ij} = c_{ij} \left\{ \frac{1}{1 + \exp\left\{-f_i - \sum_{j=1}^{N} c_{ij} x^*_{j0}\right\}} - \frac{1}{\left[1 + \exp\left\{-f_i - \sum_{j=1}^{N} c_{ij} x^*_{j0}\right\}\right]^2} \right\}$$

and Q (X) and R (X) are the remainders of the Taylor expansions containing all the nonlinear terms of the relevant functions at $x_{ik} = x^*_{ik}$. Clearly the functions Q (X) and R (X) are negligible for small X_{ik}'s. Therefore we can omit Q and R in Eqs. (III.3) for sufficiently small X_{ik}'s. Consider the linear equations obtained this way. Using the well known criterion of Routh-Hurwitz and other general stability criteria, we can study the stability properties of the solutions of Eqs. (III.3) in the neighborhood of the steady-state solutions $X_{ik} = 0$, which yield naturally similar results for the original system (II.2) in the neighborhood of $\{x^*_{ik}\}$. The effect of the nonlinear terms on the first approximation can be analysed by use of asymptotic methods (cfr. [3]).

In the next section we deal with numerical solution of the system (I.1).

IV. We now consider the system (I.1) where $f_i(t)$ is not restricted to be a constant. To establish the solutions of Eqs. (I.1) numerically, we put

$$\text{(IV.1)}\qquad \begin{cases} u_i(v_i) = \dfrac{1}{1 + e^{-v_i}} \quad , \quad v_i(t) = f_i(t) + \sum_{j=1}^{K} c_{ij} x_j \\ w_{ik} = \displaystyle\int_0^t g_{ik}[x_i(s)]\, e^{-p_{ik}(t-s)}\, ds\,. \end{cases}$$

Then Eqs. (I.1) can be written as

$$\text{(IV.2)}\qquad \frac{dx_i}{dt} = a[u_i - x_i] + \sum_{k=1}^{K} b_{ik} w_{ik} \quad , \quad x_i(0) = x^0_i\,.$$

Let h be a sufficiently small positive number. Put

$$\text{(IV.3)}\qquad x_{i,r} = x_i(rh) \quad , \quad x^{(m)}_{i,r} = \left.\frac{d^m x_i}{dt^m}\right|_{t=rh} \qquad (i = 1, \cdots, N\ ;\ r = 0, 1, \cdots, R).$$

We assume that $x^0_i = x_{i,0}$. Then we can write

$$\text{(IV.4)}\qquad x_{i,r+1} = x_{i,r} + \frac{h}{1!} x'_{i,r} + \frac{h^2}{2!} x''_{i,r} + \cdots + \frac{h^M}{M!} x^{(M)}_{i,r}$$

by Taylor's formula, where we assume that

(IV.5)
$$\frac{h^{M+1}}{(M+1)!}\,|x_{i,r}^{(M+1)}| < \varepsilon\,,$$

where ε is a given sufficiently small positive number. Clearly we have

(IV.6)
$$\frac{d^m x_i}{dt^m} = a\left[\frac{d^{m-1} u_i}{dt^{m-1}} - \frac{d^{m-1} x_i}{dt^{m-1}}\right] + \sum_{k=1}^{N} b_{ik}\,\frac{d^{m-1} w_{ik}}{dt^{m-1}}\,.$$

We now put

(IV.7)
$$[n] = \frac{d^n u_i}{dv_i^n}\quad,\quad (n) = \frac{d^n v_i}{dt^n}\,,\qquad n = 1\,,2\,,\cdots,r\,.$$

Clearly

(IV.8)
$$\begin{cases} \dfrac{du_i}{dt} = [1]\,(1)\,,\\[2mm] \dfrac{d^2 u_i}{dt^2} = [2]\,(1)^2 + [1]\,(2)\,.\\ \cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

We can show by the mathematical induction that

(IV.9)
$$\frac{d^n u_i}{dt^n} = \sum_{r=1}^{n}[r]\left\{\sum_{\vec{\alpha}(n\,;\,r)\in E} a_n(r\,;\vec{\alpha}(n\,;r))\prod_{q=1}^{n}(q)^{\alpha_q(n;r)}\right\},$$

where

(IV.10)
$$\begin{cases} E = \left\{\vec{\alpha}(n\,;r)\;\middle|\;\sum_{q=1}^{n}\alpha_q(n\,;r) = r \quad\text{and}\quad \sum_{q=1}^{n} q\alpha_q(n\,;r) = r\right\},\\ \vec{\alpha}(n\,;r) = [\alpha_1(n\,;r)\,,\alpha_2(n\,;r)\,,\cdots,\alpha_n(n\,;r)]\,,\qquad \alpha_q(n\,;r)\\ \qquad\qquad \text{integer and}\quad 0\le\alpha_q(n\,;r)\le r\,, \end{cases}$$

and $\alpha_n(r\,;(n\,;r))$ is defined by the following recursive formula:

(IV.11)
$$a_n(r\,;\vec{\alpha}(n\,;r)) = \mu(r\,;\vec{\alpha}(n\,;r)) + \sum_q \nu_q(r\,;\vec{\alpha}(n\,;r))\,,$$

where

(IV.12)
$$\mu(r\,;\vec{\alpha}(n\,,r)) = \begin{cases} a_{n-1}(r-1\,;\vec{\alpha}'(n-1\,,r-1))\\ \quad\text{if}\quad \alpha'_{q'}(n-1\,,r-1) = \alpha_{q'}(n\,;r) - \delta_{q',1}\\ \quad\text{and}\quad \alpha_n(n\,,r) = 0\,,\quad q' = 1\,,2\,,\cdots,n-1\,,\\ 0\,,\quad\text{otherwise}\,, \end{cases}$$

(IV.13)
$$\nu(r\,;\vec{\alpha}(n\,;r)) = \begin{cases} \alpha''_q(n-1\,;r)\;a_{n-1}(r\,;\alpha''(n-1\,;r))\\ \quad\text{if}\quad \alpha''_{q'}(n-1\,;r) = \alpha_{q'}(n\,;r) - \delta_{q',q+1}\,,\\ \qquad q' = 1\,,\cdots,n\quad\text{with}\quad \alpha''_n(n-1\,;r) = 0\,,\\ 0,\quad\text{otherwise}\,, \end{cases}$$

and

(IV.14)
$$\delta_{q',q} = \begin{cases} 0 & \text{if } q' \neq q \\ 1 & \text{if } q' = q . \end{cases}$$

We can also prove by the mathematical induction the following formula

(IV.15)
$$\frac{d^n u_i}{dv_i^n} = u_i z_i P_n(z_i)$$

where

(IV.16)
$$z_i = u_i e^{-v_i}$$

and $P_n(\theta)$ is a polynomial of degree $n-1$ defined by the recursive formula

(IV.17)
$$P_1(\theta) = 1 \quad , \quad P_n(\theta) = (2\theta - 1) P_{n-1}(\theta) + \theta (n-1) P'_{n-1}(\theta)$$
$$n = 2, 3, 4, \cdots .$$

Further, we have

(IV.18)
$$\frac{d^n v_i}{dt^n} = \frac{d^n f_i}{dt^n} + \sum_{j=n}^{N} c_{ij} \frac{d^n x_i}{dt^n} .$$

and

(IV.19)
$$\frac{d^n w_{i,k}}{dt^n} = \frac{d^{n-1} x_i}{dt^{n-1}} - p_{ik} \frac{d^{n-1} w_{i,k}}{dt^{n-1}} ,$$

by virtue of Eqs. (IV.1). We now put

(IV.20)
$$w_{i,k;r} = \int_0^{rh} g_{ik}[x_i(s)]\, e^{-p_{ik}(rh-s)}\, ds .$$

We can show, by a quadratic approximation, that

(IV.21)
$$w_{i,k;r} = e^{-p_{ik}h} \left[w_{i,k;r-1} - \frac{1}{p_{ik}} \left\{ g_{i,k;r-1} - \frac{1}{p_{ik}} g'_{i,k;r-1} + \frac{2\gamma}{p_{ik}^2} \right\} \right] +$$
$$+ \frac{1}{p_{ik}} \left[g_{i,k;r} - \frac{1}{p_{ik}} g'_{i,k;r} + \frac{2\gamma}{p_{ik}^2} \right] ,$$

where

(IV.22)
$$g_{i,k;r} = g_{ik}[x_i(rh)] \quad , \quad g'_{i,k;r} = \left. \frac{dg_{ik}(x_i)}{dx_i} \frac{dx_i}{ds} \right|_{s=rh}$$

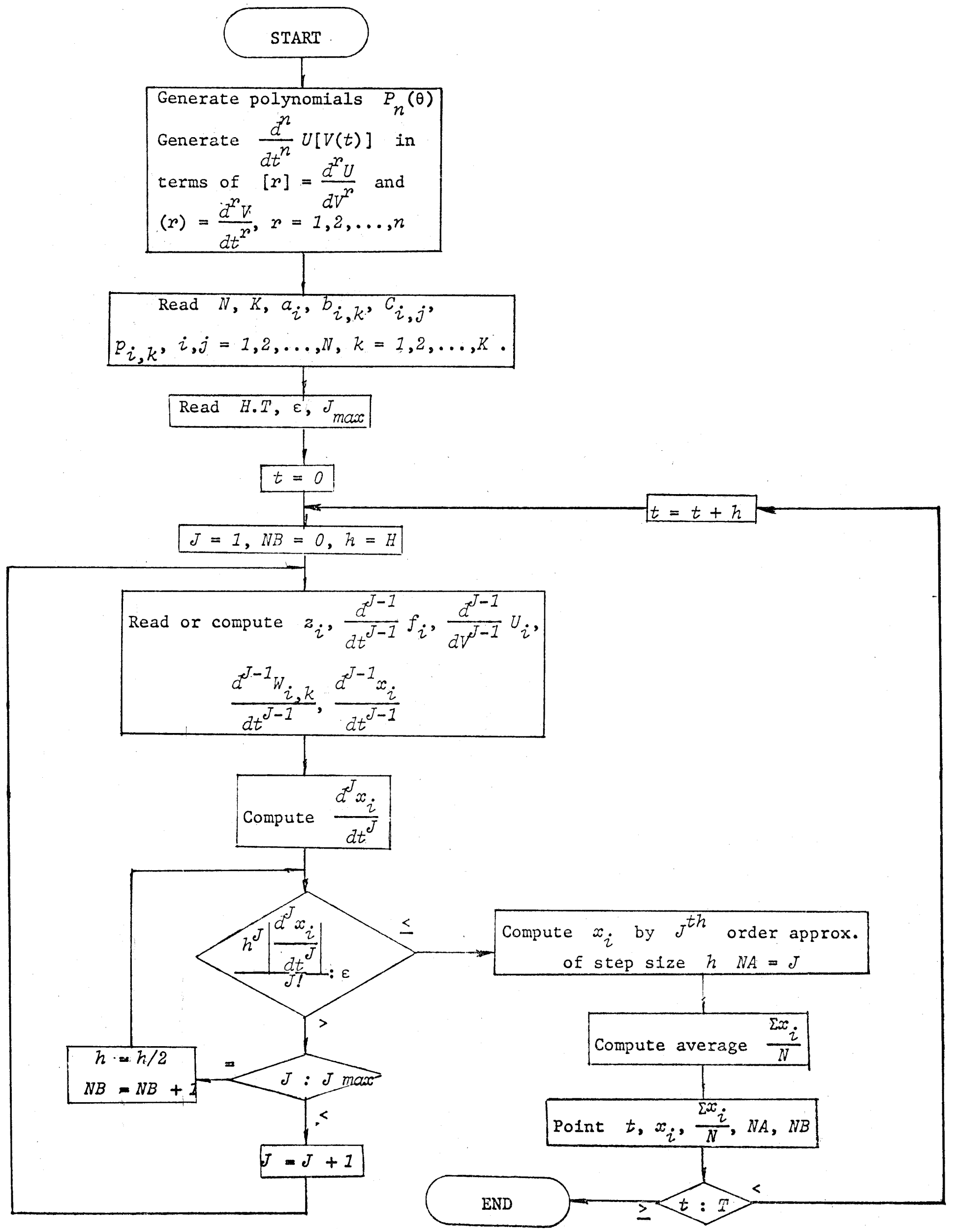
and

$$\gamma = \frac{1}{h^2} [g_{i,k;r} - g_{i,k;r-1} - h g'_{i,k;r-1}] .$$

Combining formula (IV.6) with (IV.9) and (IV.19), we obtain the necessary recurrence relationship to construct the solutions $x_i(t)$ step by step.

The flowchart of the above computation is given below. The corresponding computer programming can be made easily. In our investigations we observed oscillatory, non-oscillatory, damped and undamped solutions depending on the choice of the parameters and on the form of the functions $f_i(t)$ and $g_{ik}(x_i)$.

FLOWCHART FOR EQS (1.1)



In a forthcoming paper we shall investigate the effect of the nonlinear terms $Q_i(X)$ and $R_{ik}(X)$ on the behaviour of the first approximate solutions.

V. The authors gratefully acknowledge the use of the computing facilities of the University of Alberta, and are pleased to emphasize once more that the invaluable research work of Professor M. Picone [4] and his Institute (INAC) constitutes a permanent example of how to embark upon, and how to solve, difficult problems arising from the study of natural phenomena.

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