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On the strong solutions of the Navier-Stokes equations in three dimensional space. Nota II

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Analisi matematica. — *On the strong solutions of the Navier–Stokes equations in three dimensional space* (*). Nota II di GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si danno le dimostrazioni dei Teoremi 1 e 2 enunciati nella Nota I.

3. – PROOF OF THEOREMS 1 AND 2

Setting

$$M = \max_{x \in \overline{\Omega}} |\vec{z}(x)| \tag{3.1}$$

let us denote by $\vec{v}(x, t)$ the solution a.e. in Q of the Navier–Stokes inequalities relative to K_M (with $\vec{f} = 0$) satisfying the initial and boundary conditions (1.2), (1.3); by the theorem proved in § 2, such a solution exists and is unique.

We begin by proving the existence theorem; we shall show, for this, that $\vec{v}(x, t)$ is also a solution a.e. in Q of the homogeneous Navier–Stokes equations satisfying (1.2), (1.3).

By the assumptions made, $\vec{v}(t)$ is such that

$$\vec{v}(t) \in L^2(0, T; N^1), \quad \vec{v}'(t) \in L^2(0, T; N^1), \quad \Delta\vec{v}(t) \in L^2(0, T; N^0), \tag{3.2}$$
$$\vec{v}(t) \in K_M \quad \text{in } [0, T]$$

and, $\forall \vec{l}(t) \in L^2(0, T; N^0_{K_M})$,

$$\int_0^T \{(\vec{v}'(t) - \mu\Delta\vec{v}, \vec{v}(t) - \vec{l}(t))_{N^0} + b(\vec{v}(t), \vec{v}(t), \vec{v}(t) - \vec{l}(t))\} \, dt \leq 0. \tag{3.3}$$

Denote now by Q′ the set ⊂ Q in which $|\vec{v}| = M$ and by Q″ the set Q — Q′; bearing in mind what was proved in § 1, the function $\vec{v}(x, t)$ is a solution of the Navier–Stokes equations a.e. in Q″. Setting in (3.3)

$$\vec{l}(x, t) = \begin{cases} \vec{v}(x, t) & \text{when } (x, t) \in Q'' \\ 0 & \text{when } (x, t) \in Q' \end{cases} \tag{3.4}$$

we obtain then

$$\int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v}\,\text{grad})\,\vec{v}\right) \times \vec{v} \, dQ' \leq 0. \tag{3.5}$$

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(**) Nella seduta del 10 febbraio 1973.

Observe that, by the definitions given,

$$\frac{\partial \vec{v}}{\partial t} \in L^2(Q) \quad , \quad \frac{\partial^2 \vec{v}}{\partial x_j^2} \in L^2(Q) \qquad (j = 1, 2, 3). \tag{3.6}$$

Moreover, since $|\vec{v}|$ takes its maximum value on Q′,

$$-\int_{Q'} \Delta\vec{v} \times \vec{v}\, dQ' = -\int_{Q'} \sum_{j,k=1}^{3} \frac{\partial^2 v_j}{\partial x_k^2} v_j\, dQ' = \tag{3.7}$$

$$= \int_{Q'} \sum_{j,k=1}^{3} \left[\frac{\partial}{\partial x_k}\left(\frac{\partial v_j}{\partial x_k} v_j\right) - \left(\frac{\partial v_j}{\partial x_k}\right)^2 \right] dQ' =$$

$$= \int_{Q'} \sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k}\right)^2 dQ' - \int_{Q'} \sum_{k=1}^{3} \frac{\partial^2}{\partial x_k^2}\left(\frac{1}{2}|\vec{v}|^2\right) dQ' \geq \int_{Q'} \sum_{j,k=1}^{3} \left(\frac{\partial v_j}{\partial x_k}\right)^2 dQ' \geq 0$$

$$\int_{Q'} \frac{\partial \vec{v}}{\partial t} \times \vec{v}\, dQ' = \frac{1}{2}\int_{Q'} \frac{\partial}{\partial t}\left(\frac{1}{2}|\vec{v}|^2\right) dQ' = 0 \tag{3.8}$$

$$\int_{Q'} (\vec{v} \times \text{grad})\, \vec{v} \times \vec{v}\, dQ' = \int_{Q'} (\text{rot}\, \vec{v}) \wedge \vec{v} \times \vec{v}\, dQ' + \frac{1}{2}\int_{Q'} \text{grad}\, |\vec{v}|^2 \times \vec{v}\, dQ' = 0. \tag{3.9}$$

Hence, substituting (3.8), (3.9) into (3.5), we obtain

$$-\int_{Q'} \Delta\vec{v} \times \vec{v}\, dQ' \leq 0$$

and consequently, by (3.7),

$$\int_{Q'} \Delta\vec{v} \times \vec{v}\, dQ' = 0\,. \tag{3.10}$$

Let us now set in (3.3)

$$\vec{l}(x, t) = \begin{cases} \vec{v}(x, t) & \text{when } (x, t) \in Q'' \\ \vec{h}(x, t) & \text{when } (x, t) \in Q' \end{cases}$$

where $\vec{h}(x, t)$ is the restriction to Q′ of a function $\in L^2(0, T; N^0_{K_M})$; we obtain

$$\int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v} \times \text{grad})\, \vec{v}\right) \times (\vec{v} - \vec{h})\, dQ' \leq 0 \tag{3.11}$$

and consequently, by (3.8), (3.9), (3.10),

$$\int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v} \times \text{grad})\, \vec{v}\right) \times \vec{h}\, dQ' \geq 0. \tag{3.12}$$

Since we can change $\vec{h}$ in $-\vec{h}$, it follows from (3.12) that

$$\int_{Q'} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v}\times\text{grad})\,\vec{v}\right) \times \vec{h}\, dQ' = 0 . \tag{3.13}$$

$\forall \vec{h}$ restriction to Q' of a function $\in L^2(0, T; N^0_{K_M})$ and therefore also, multiplying (3.13) by a suitable constant, $\forall \vec{h} \in N^0(Q')$.

Observe that, as already pointed out, $\vec{v}(x, t)$ is a solution of the Navier–Stokes equations a.e. in Q'' and satisfies therefore the equation

$$\int_{Q''} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v}\times\text{grad})\,\vec{v}\right) \times \vec{h}\, dQ'' = 0 . \tag{3.14}$$

$\forall \vec{h} \in N^0(Q'')$. Hence, by (3.13), (3.14),

$$\int_{Q} \left(\frac{\partial \vec{v}}{\partial t} - \mu\Delta\vec{v} + (\vec{v}\times\text{grad})\,\vec{v}\right) \times \vec{h}\, dQ = 0 . \tag{3.15}$$

$\forall \vec{h} \in N^0(Q)$, i.e. $\vec{v}(x, t)$ is a solution a.e. in Q of the Navier–Stokes equations. The existence of a solution a.e. in Q satisfying (1.2), (1.3) is therefore proved.

The uniqueness of such a solution is, on the other hand, well known (see Prodi [6]; refer to the bibliography at the end of Note I).

Let us now prove the maximum principle expressed by Theorem 2.

This follows directly from the proof of Theorem 1. We have, in fact, shown that the solution a.e. in Q of the homogeneous Navier–Stokes equations belongs to K_M, i.e. that

$$|\vec{v}(x, t)| \leq M \qquad \text{a.e. in } Q ,$$

where

$$M = \max_{x \in \overline{\Omega}} |\vec{z}(x)| .$$

Hence

$$\max_{(x,t) \in \overline{Q}} |\vec{v}(x, t)| = \max_{x \in \overline{\Omega}} |\vec{v}(x, 0)| ,$$

which proves the maximum principle.