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On torsion in unitary K-theory of compact Lie groups

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 14 aprile 1973

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On torsion in unitary* K*–theory of compact Lie groups*. Nota di RENÉ P. HELD e ULRICH SUTER, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Adoperando la sequenza spettrale di Atiyah–Hirzebruch, si determina la K–teoria unitaria per un gruppo compatto di Lie con gruppo fondamentale ciclico di ordine primo. Questo risultato viene poi applicato al calcolo della K–teoria dei gruppi simplettici proiettivi.

1. INTRODUCTION

The unitary K–theory of a compact simply connected Lie group G is known to be an exterior algebra the generators of which can be given by means of the basic irreducible complex representations of G (see [4]).

In case the compact Lie group G is not simply connected one essentially faces the problem of describing the torsion part of $K^*(G)$.

We shall give a complete answer to that problem in the case of a compact, connected Lie group G with finite fundamental group $\pi_1(G)$ being cyclic of prime order. In fact the torsion part of $K^*(G)$ turns out to be expressible in terms of the complex representation rings of $\pi_1(G)$ and $\tilde{G}$, the universal covering group of G.

The outline of the proof of the main theorem is followed by applications.

2. THE MAIN RESULT

Suppose G is a compact connected Lie group and assume that its fundamental group $\pi_1(G) = \pi$ is finite. Further, let $i : \pi \hookrightarrow \tilde{G}$ be the inclusion map of the (Deck-transformation) group π into the universal covering group

(*) Nella seduta del 14 aprile 1973.

$\tilde{G}$ of G. Consider the induced map $i^* : R(\tilde{G}) \to R(\pi)$ of the complex representation rings and denote by $T(G)$ the quotient $R(\pi)/(i^*(I_{\tilde{G}}))$ and by $\tilde{T}(G)$ the direct summand of T such that $T \cong Z \oplus \tilde{T}$.

THEOREM. *Let* G *be of rank* k *and* $\pi_1(G) \cong Z_p$ *where* p *is a prime. Then there exist elements* $\nu_1, \nu_2, \cdots, \nu_{k-1}, \varepsilon_k \in K^1(G)$ *such that as rings*

$$K^*(G) \cong \Lambda_Z(\nu_1, \cdots, \nu_{k-1}, \varepsilon_k) \otimes T(G)/(\varepsilon_k \otimes \tilde{T}(G)).$$

3. Outline of the proof

The theorem is a consequence of the lemmas which appear below. In fact to a certain extent we synthesize the reasoning the authors used to compute the K–theory of the special orthogonal groups (see [3]).

Let $u : \tilde{G} \to G$ be the universal covering and—according to [4]—let $\lambda_1, \cdots, \lambda_k \in K^1(\tilde{G})$ be generators of $K^*(\tilde{G}) = \Lambda_Z(\lambda_1, \cdots, \lambda_k)$. Then the "free part" of $K^*(G)$ is given as follows.

LEMMA 1. *There exist elements* $\nu_1, \nu_2, \cdots, \nu_{k-1}, \varepsilon_k$ *in* $K^1(G)$ *such that*

(i) $u^*(\nu_1) = \lambda_1, \cdots, u^*(\nu_{k-1}) = \lambda_{k-1}$ *and* $\varepsilon_k = u_*(\lambda_k)$ *(transfer) with* $u^*(\varepsilon_k) = p\lambda_k$;

(ii) $K^*(G)$ *mod Tors.* $K^*(G)$ *is the exterior algebra* $\Lambda_Z(\bar{\nu}_1, \cdots, \bar{\nu}_{k-1}, \bar{\varepsilon}_k)$ *generated by the projections* $\bar{\nu}_1, \cdots, \bar{\varepsilon}_k$ *of the elements* $\nu_1, \cdots, \nu_{k-1}, \varepsilon_k$;

(iii) $\nu_1, \cdots, \nu_{k-1}, \varepsilon_k$ *generate an exterior algebra* $\Lambda_Z(\nu_1, \cdots, \nu_{k-1}, \varepsilon_k)$ *in* $K^*(G)$.

This Lemma is proven by using properties of the Chern character (see [2]) and a theorem of Hodgkin's [4; (2.2)] concerning Z_2–graded Hopf algebras.

The "torsion part" of $K^*(G)$ is obtained by analyzing the Atiyah-Hirzebruch spectral sequences $E(\Lambda)$ and $E(\Lambda_n)$ of the fibre bundle $\Lambda = (\tilde{G} \underset{u}{\to} G \underset{q}{\to} B_\pi)$ and its restrictions Λ_n to the n–skeleton of the classifying space B_π (q denotes the classifying map). Let s be the greatest integer such that $E(\Lambda_{2s-1})$ is trivial. Then the pertinent information we get from looking at these spectral sequences can be summarized as follows.

LEMMA 2. (i) *Let* $G_{2s-2} = q^{-1}(B_\pi^{2s-2})$, $q_{2s-2} = q \mid G_{2s-2}$ *and* j_{2s-2}: $G_{2s-2} \hookrightarrow G$ *the canonical inclusion. Then in the commutative diagram*

$$\begin{array}{ccc} K^*(G) & \xrightarrow[j^*_{2s-2}]{} & K^*(G_{2s-2}) \\ \uparrow {\scriptstyle q^*} & & \uparrow {\scriptstyle q^*_{2s-2}} \\ K^*(B_\pi) & \longrightarrow & K^*(B_\pi^{2s-2}) \end{array}$$

the homomorphisms j^*_{2s-2} *and* q^*_{2s-2} *are injective*;

(ii) $|\text{Tors. } K^*(G)| \leq p^{(s-1)2^{k-1}}$.

From this lemma we get a complete description of the ring $K^*(G)$. Let $T = \text{im}\{K^0(B_\pi) \underset{q^*}{\to} K^0(G)\} \cong K^0(B_\pi^{2s-2})$ and $\tilde{T}$ such that $T \cong Z \oplus \tilde{T}$.

COROLLARY 3. *We have the following ring isomorphism*

$$K^*(G) \cong \Lambda_Z(\nu_1, \cdots, \nu_{k-1}, \varepsilon_k) \otimes T/(\varepsilon_k \otimes \tilde{T}) .$$

Eventually we are able to identify T with the quotient $R(\pi)/(i^* I_{\tilde{G}})$ by using the "α–construction", i.e., a natural homomorphism $\alpha : R(\pi) \to K^0(B_\pi)$ defined by Atiyah [1] and a result of Segal [5] saying that the I_π–adic topology on $R(\pi)$ coincides with the $I_{\tilde{G}}$–adic topology induced by $i^* : R(\tilde{G}) \to R(\pi)$. Hence the theorem is established.

4. APPLICATIONS

(4.1) *The K–theory of the projective symplectic groups* $PSp(n)$.

The center of the symplectic group $Sp(n)$ is cyclic of order 2. The complex representation ring of $Sp(n)$ is a polynomial algebra $Z[\lambda_1, \cdots, \lambda_n]$, where λ_j is the j–th exterior power of the canonical representation $\lambda_1 : Sp(n) \to U(2n)$. Furthermore we have $R(Z_2) = Z[\rho]/(\rho^2 - 1)$, where $\rho : Z_2 \to U(1)$ denotes the canonical representation of Z_2. The homomorphism $i^* : R(Sp(n)) \to R(Z_2)$ induced by the inclusion $i : Z_2 \to Sp(n)$ is given by $i^*(\lambda_j) = \binom{2n}{j} \rho^j$, for $j = 1, 2, \cdots, n$. Hence the ideal $J = (i^* I_{Sp(n)}) \subset I_{Z_2}$ is given by $J = (2^{\nu_2(n)+1} \cdot (\rho - 1))$, where $2^{\nu_2(n)}$ is the highest power of 2 dividing n. We therefore have the following description of $K^*(PSp(n))$.

PROPOSITION. *There are elements* $\nu_1, \cdots, \nu_{n-1}, \varepsilon_n \in K^1(PSp(n))$ *and* $\xi \in K^*(PSp(n))$ *such that*

(i) $K^*(PSp(n)) \cong \Lambda_Z(\nu_1, \cdots, \nu_{n-1}, \varepsilon_n) \otimes T/(\varepsilon_n \otimes \xi)$ *where* T *is the subring of* $K^*(PSp(n))$ *generated by* 1 *and* ξ,

(ii) *The element* ξ *is subject to the relations*

$$\xi^2 + 2\xi = 0 \quad , \quad 2^{\nu_2(n)+1} \xi = 0 .$$

(4.2) REMARK. By means of the main theorem it is easy to compute the K–theory for Lie groups such as $SO(n)$, $PU(n)$, PE_6 and PE_7.

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