Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On Almansi problem for an elastic orthotropic cylinder

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.3, p. 441–446.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_3_441_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1973.

Meccanica. — *On Almansi problem for an elastic orthotropic cylinder.* Nota di CONSTANTIN I. BORŞ, presentata (*) dal Socio D. GRAFFI.

RIASSUNTO. — Si dà una soluzione per il problema di Almansi definito da (1) e (2). La presente soluzione qui è più semplice di quelle fin'ora conosciute.

We shall consider an orthotropic cylinder limited by two planes $x_3 = 0$, $x_3 = h\,(h > 0)$ and by the surface $\mathfrak{F}$.

Let $\mathfrak{D}$ be the domain of the cylinder and S the domain and also the area of a cross-section of the cylinder. We shall denote by Γ the boundary of S.

We take the x_3-axis to be central-line of the cylinder and the axes x_1 and x_2 to be principal axes of inertia of the end $x_3 = 0$. In this case we have

$$\iint_S x_1 \, d\sigma = 0 \quad , \quad \iint_S x_2 \, d\sigma = 0 \quad , \quad \iint_S x_1 x_2 \, d\sigma = 0 .$$

We suppose that there are no body forces so that the equilibrium equations can be written in the form

$$\sigma_{ij,j} = 0 \qquad \text{in} \quad \mathfrak{D} \;^{(1)}. \tag{1}$$

We suppose also that the tractions applied on the lateral surface are given by

$$\sigma_{ij}\, n_j = \sum_{k=0}^{n} \tau_i^{(k)} (x_1 , x_2)\, x_3^k \qquad \text{on} \quad \mathfrak{F}, \tag{2}$$

where the functions $\tau_i^{(k)}$ are given and n_i are the direction cosines of the exterior normal to $\mathfrak{F}$.

At the ends there are applied tractions in order to equilibrate the loads (2).

The problem defined by (1) and (2) is known as Almansi's problem.

Here, we shall find a solution which satisfies the equations (1) and (2). After that it remains to satisfy the end conditions, but this is the Saint–Venant problem and we know how to solve it [2].

Some methods are known to solve the Almansi problem, for the isotropic case [1], [4] and for the orthotropic case [5]. In the paper [1] and [5] there are started with some representations of the stresses and in the paper [4] it is started with a representation of the displacements.

Generally, in the known solutions a step by step method is given which allows us to solve the problem having the tractions $\tau_i(x_1 , x_2)\, x_3^{s+1}$ on the lateral

(*) Nella seduta del 10 marzo 1973.

(1) We use the summation convention over repeated indices. The index j after comma indicates partial differentiation with respect to x_j.

surface when the solution for the tractions $\tau_i(x_1, x_2)\, x_3^s$ is known. Knowing the solution for $s = 0$ we can obtain the solution for every s. This process must be performed for each partial traction $\tau_i^{(k)} x_3^k$ $(k = 0, 1, 2, \cdots, n;$ not summed). The final solution is obtained by superposition of solutions of the partial problems.

Here, a solution of the complete problem is given such that its determination is equivalent to the calculus of the first step in the other known solutions.

The relations between the stress components σ_{ij} and the strain components γ_{ij} are taken in the form

$$
\text{(3 a)} \quad \begin{cases} \sigma_{11} = A\gamma_{11} + H\gamma_{22} + G\gamma_{33} \quad , \quad \sigma_{22} = H\gamma_{11} + B\gamma_{22} + F\gamma_{33} , \\ \sigma_{33} = G\gamma_{11} + F\gamma_{22} + C\gamma_{33} , \\ \sigma_{23} = L\gamma_{23} \quad , \quad \sigma_{31} = M\gamma_{31} \quad , \quad \sigma_{12} = D\gamma_{12} \end{cases}
$$

and the inverse relations of (3 a) in the form

$$
\text{(3 b)} \quad \begin{cases} \gamma_{11} = \dfrac{1}{E}(\nu_{11}\sigma_{11} + \nu_{12}\sigma_{22} - \nu_1\sigma_{33}) \quad , \quad \gamma_{22} = \dfrac{1}{E}(\nu_{12}\sigma_{11} + \nu_{22}\sigma_{22} - \nu_2\sigma_{33}), \\ \gamma_{33} = \dfrac{1}{E}(-\nu_1\sigma_{11} - \nu_2\sigma_{22} + \sigma_{33}), \\ \gamma_{23} = \dfrac{1}{L}\sigma_{23} \quad , \quad \gamma_{31} = \dfrac{1}{M}\sigma_{31} \quad , \quad \gamma_{12} = \dfrac{1}{D}\sigma_{12} , \end{cases}
$$

where $A, B, \cdots, E, \nu_{ij}, \nu_i$ are constants which characterise the elastic properties of material of the cylinder.

We shall try to solve the Almansi problem supposing that the displacement components u_i are given by

$$
\text{(4)} \quad \begin{cases} u_1 = \displaystyle\sum_{k=0}^{n+1} \left(\Theta_1^{(k)} + a_k \frac{x_3^2}{(k+3)(k+2)} \right) x_3^{k+1} + \sum_{k=0}^{n} \varphi_k x_3^k , \\ u_2 = \displaystyle\sum_{k=0}^{n+1} \left(\Theta_2^{(k)} + b_k \frac{x_3^2}{(k+3)(k+2)} \right) x_3^{k+1} + \sum_{k=0}^{n} \psi_k x_3^k , \\ u_3 = -\displaystyle\sum_{k=0}^{n+1} \frac{1}{k+2}(a_k x_1 + b_k x_2 - A_k)\, x_3^{k+2} + \sum_{k=0}^{n+1} \omega_k x_3^k , \end{cases}
$$

where

$$
\Theta_1^{(k)} = \frac{1}{2} a_k (\nu_1 x_1^2 - \nu_2 x_2^2) + b_k \nu_1 x_1 x_2 - c_k x_2 - A_k \nu_1 x_1 ,
$$

$$
\Theta_2^{(k)} = a_k \nu_2 x_1 x_2 + \frac{1}{2} b_k (\nu_2 x_2^2 - \nu_1 x_1^2) + c_k x_1 - A_k \nu_2 x_2 ,
$$

$\varphi_k, \psi_k, \omega_k$ are unknown functions and a_k, b_k, c_k, A_k are constants which must be determined.

We like to remark that the additional terms in the equations (4) have some mechanical meaning.

If we denote by $\tau_{ij}^{(k)}$ the stresses corresponding to the displacements $u_1^{(k)} = \varphi_k$, $u_2^{(k)} = \psi_k$, $u_3^{(k)} = 0$, then the stresses σ_{ij} corresponding to the displacements (4) are given by

$$\sigma_{11} = \sum_{k=0}^{n} \tau_{11}^{(k)} x_3^k + G \sum_{k=0}^{n+1} k\omega_k x_3^{k-1}, \quad \sigma_{22} = \sum_{k=0}^{n} \tau_{22}^{(k)} x_3^k + F \sum_{k=0}^{n+1} k\omega_k x_3^{k-1},$$

$$\sigma_{33} = \sum_{k=0}^{n} \tau_{33}^{(k)} x_3^k - E \sum_{k=0}^{n+1} (a_k x_1 + b_k x_2 - A_k) x_3^{k+1} + C \sum_{k=0}^{n+1} k\omega_k x_3^{k-1},$$

$$\sigma_{23} = L \left\{ \sum_{k=0}^{n+1} [(k+1)\,\Theta_2^{(k)} + \omega_{k,2}]\, x_3^k + \sum_{k=0}^{n} k\psi_k x_3^{k-1} \right\},$$

$$\sigma_{31} = M \left\{ \sum_{k=0}^{n+1} [(k+1)\,\Theta_1^{(k)} + \omega_{k,1}]\, x_3^k + \sum_{k=0}^{n} k\varphi_k x_3^{k-1} \right\},$$

$$\sigma_{12} = \sum_{k=0}^{n} \tau_{12}^{(k)} x_3^k .$$

Using the above formulae, from the third equation (1), we get

(5 a) $$\Delta\omega_k = -(k+1)\,[(M\nu_1 + L\nu_2 - E)(a_k x_1 + b_k x_2 - A_k) + \tau_{33}^{(k)} + M\varphi_{k+1,1} + L\psi_{k+1,2} + (k+2)\,C\omega_{k+2}] \quad \text{in } S$$

$$(k = 0, 1, 2, \cdots, n),$$

where

$$\Delta = M \frac{\partial^2}{\partial x_1^2} + L \frac{\partial^2}{\partial x_2^2} .$$

The third condition (2) will be satisfied if

(5 b) $$\mathfrak{D}\omega_k = \tau_3^{(k)} - (k+1)\,[M(\Theta_1^{(k)} + \varphi_{k+1})\, n_1 + L(\Theta_2^{(k)} + \psi_{k+1})\, n_2] \quad \text{on } \Gamma,$$

where the operator $\mathfrak{D}$ is given by

$$\mathfrak{D} = M n_1 \frac{\partial}{\partial x_1} + L n_2 \frac{\partial}{\partial x_2} .$$

The condition of existence of the function ω_k requires that

(6) $$\int_\Gamma \mathfrak{D}\omega_k \, ds - \iint_S \Delta\omega_k \, d\sigma = 0.$$

It is easy to see that the function ω_k exists with arbitrary a_k, b_k, c_k. From the condition (6) we obtain

(7) $$ESA_k = -\frac{1}{k+1} \int_\Gamma \tau_3^{(k)} \, ds - \iint_S [\tau_{33}^{(k)} + (k+2)\, C\omega_{k+2}]\, d\sigma$$

$$(k = 0, 1, 2, \cdots, n+1).$$

The first two equations of equilibrium (1) give

$$
\text{(8 a)}\quad \begin{cases} \tau_{11,1}^{(k)} + \tau_{12,2}^{(k)} + (k+1)\,[(\mathrm{G}+\mathrm{M})\,\boldsymbol{\omega}_{k+1,1} + (k+2)\,\mathrm{M}\,(\Theta_1^{(k+1)} + \varphi_{k+2})] = 0, \\ \tau_{12,1}^{(k)} + \tau_{22,2}^{(k)} + (k+1)\,[(\mathrm{F}+\mathrm{L})\,\boldsymbol{\omega}_{k+1,2} + (k+2)\,\mathrm{L}\,(\Theta_2^{(k+1)} + \psi_{k+2})] = 0 \quad \text{in S}, \\ \qquad (k = 0, 1, \cdots, n) \end{cases}
$$

and from the first two equations (2) we have

$$
\text{(8 b)}\quad \begin{cases} \tau_{11}^{(k)} n_1 + \tau_{12}^{(k)} n_2 = \tau_1^{(k)} - (k+1)\,\mathrm{G}\boldsymbol{\omega}_{k+1}\, n_1, \\ \tau_{12}^{(k)} n_1 + \tau_{22}^{(k)} n_2 = \tau_2^{(k)} - (k+1)\,\mathrm{F}\boldsymbol{\omega}_{k+1}\, n_2 \qquad \text{on } \Gamma. \end{cases}
$$

Everywhere, in this paper, we must remember that [see (4)]

$$
\text{(9)}\qquad \boldsymbol{\omega}_{n+1+s} = \varphi_{n+s} = \psi_{n+s} = \tau_{ij}^{(n+s)} = \tau_i^{(n+s)} = 0 \qquad (s = 1, 2, \cdots).
$$

The equations (8 a) and (8 b) define a plan problem with body forces, for the stresses $\tau_{\alpha\beta}^{(k)}$ $(\alpha, \beta = 1, 2;\ k = 0, 1, 2, \cdots, n)$.

The solution of the problem (8 a), (8 b) gives us the stress components $\tau_{\alpha\beta}^{(k)}$ and also the functions φ_k, ψ_k because these functions are the component of displacements in this problem.

Now, let us examine the existence of the solution of the problem (8 a), (8 b). For that purpose let $\overset{*}{\tau}{}_{\alpha\beta}^{(k)}$ be a solution of the equations (8 a) and let us put down

$$
\tau_{\alpha\beta}^{(k)} = \tilde{\tau}_{\alpha\beta}^{(k)} + \overset{*}{\tau}{}_{\alpha\beta}^{(k)}.
$$

The new components $\tilde{\tau}_{\alpha\beta}^{(k)}$ will satisfy the equations

$$
\text{(10 a)}\qquad \tilde{\tau}_{\alpha\beta,\beta}^{(k)} = 0 \quad \text{in S}
$$

and the boundary conditions

$$
\text{(10 b)}\qquad \tilde{\tau}_{\alpha\beta}^{(k)}\, n_\beta = p_\alpha^{(k)} \quad \text{on } \Gamma \qquad (\alpha, \beta = 1, 2;\ k = 0, 1, \cdots, n),
$$

where

$$
p_1^{(k)} = \tau_1^{(k)} - (k+1)\,\mathrm{G}\boldsymbol{\omega}_{k+1}\, n_1 - \overset{*}{\tau}{}_{1\alpha}^{(k)}\, n_\alpha,
$$

$$
p_2^{(k)} = \tau_2^{(k)} - (k+1)\,\mathrm{F}\boldsymbol{\omega}_{k+1}\, n_2 - \overset{*}{\tau}{}_{2\alpha}^{(k)}\, n_\alpha.
$$

The necessary and sufficient conditions to solve the problem (10 a), (10 b) are given by

$$
\text{(11)}\qquad \int_\Gamma p_\alpha^{(k)}\, ds = 0 \quad (\alpha = 1, 2), \qquad \int_\Gamma (x_1 p_2^{(k)} - x_2 p_1^{(k)})\, ds = 0.
$$

These conditions allow us to determine the constants a_k, b_k, c_k before knowing the function $\boldsymbol{\omega}_k$. For that purpose we shall take into account the fact that for any function ψ which satisfies the equation $\Delta\psi = 0$ we have [3]

$$\int_{\Gamma} \omega_k \mathfrak{D}\psi \, ds = \int_{\Gamma} \psi \, \mathfrak{D}\omega_k \, ds - \iint_{S} \psi \Delta\omega_k \, d\sigma$$

and from this we can derive the following formulae

$$(12) \quad \begin{cases} \displaystyle\int_{\Gamma} M\omega_k n_1 \, ds = \int_{\Gamma} x_1 \mathfrak{D}\omega_k \, ds - \iint_{S} x_1 \Delta\omega_k \, d\sigma, \\ \displaystyle\int_{\Gamma} L\omega_k n_2 \, ds = \int_{\Gamma} x_2 \mathfrak{D}\omega_k \, ds - \iint_{S} x_2 \Delta\omega_k \, d\sigma, \\ \displaystyle\int_{\Gamma} (Mx_2 n_1 - Lx_1 n_2) \, \omega_k \, ds = \int_{\Gamma} \varphi \, \mathfrak{D}\omega_k \, ds - \iint_{S} \varphi\Delta\omega_k \, d\sigma. \end{cases}$$

Here φ is the function of torsion defined by

$$\Delta\varphi = 0 \quad \text{in} \quad S,$$

$$\mathfrak{D}\varphi = Mx_2 n_1 - Lx_1 n_2 \quad \text{on} \quad \Gamma.$$

Using the formulae (12), from the conditions (11), finally we obtain

$$(13) \quad \begin{cases} \displaystyle EI_{22} \, a_{k+1} = \frac{1}{(k+1)(k+2)} \int_{\Gamma} [\tau_1^{(k)} + (k+1) \, x_1 \, \tau_3^{(k+1)}] \, ds + \\ \displaystyle\qquad + \iint_{S} [\tau_{33}^{(k+2)} + (k+3) \, C\omega_{k+3}] \, x_1 \, d\sigma, \\ \displaystyle EI_{11} \, b_{k+1} = \frac{1}{(k+1)(k+2)} \int_{\Gamma} [\tau_2^{(k)} + (k+1) \, x_2 \, \tau_3^{(k+1)}] \, ds + \\ \displaystyle\qquad + \iint_{S} [\tau_{33}^{(k+2)} + (k+3) \, C\omega_{k+3}] \, x_2 \, d\sigma, \\ \displaystyle D_t \, c_{k+1} = \frac{1}{(k+1)(k+2)} \int_{\Gamma} [\tau_1^{(k)} x_2 - \tau_2^{(k)} x_1 + (k+1) \, \varphi\tau_3^{(k+1)}] \, ds + \\ \displaystyle\qquad + \iint_{S} \{\varphi \, (a_{k+1} x_1 + b_{k+1} x_2 - A_{k+1}) + \tau_{33}^{(k+2)} + (k+3) \, C\omega_{k+3} + \\ \displaystyle\qquad + M \, (\varphi_{,1} - x_2) \, (\widetilde{\Theta}_2^{(k+1)} + \varphi_{k+2}) - L \, (\varphi_{,2} + x_1) \, (\widetilde{\Theta}_1^{(k+1)} + \psi_{k+2})\} \, d\sigma \\ \qquad\qquad (k = 0, 1, 2, \cdots, n), \end{cases}$$

where

$$I_{\alpha\alpha} = \iint_S x_\beta^2 \, d\sigma \qquad (\alpha, \beta = 1, 2 ; \alpha \neq \beta; \text{ not summed}),$$

$$D_t = \iint_S (Lx_1^2 + Mx_2^2 + Lx_1 \varphi_{,2} - Mx_2 \varphi_{,1}) \, d\sigma > 0 \qquad (\text{see [2]}),$$

$\widetilde{\Theta}_i^{(k)}$ is obtained from $\Theta_i^{(k)}$ by dropping out the term of c_k.

It is easy to see that we must perform the calculus in the following order: taking into account (9) we can determine the constants A_{n+1} and $a_{n+1}, b_{n+1}, c_{n+1}$ from the formulae (7) and (13); the function ω_{n+1} solving the boundary value problem (5 a), (5 b) for $k = n + 1$; the functions φ_n, ψ_n solving the problem (8 a), (8 b) for $k = n$. After that we determine $A_n, a_n, b_n, c_n, \omega_n, \varphi_{n-1}, \psi_{n-1}$ and so on.

For the problem (1), (2) the constants a_0, b_0, c_0 may be taken as zero, but they can be utilized in order to satisfy some end conditions.

In this way the problem of Almansi (1), (2) is solved.

Some remarks:

– Making $n = 0$ in the above results we obtain the results concerning the Almansi-Michell problem, so that this method had unified both the problems.

– The above results can be generalized for the case when the material of the cylinder had one plane of elastic symmetry.

– We can particularize the results in order to obtain the isotropic case.

References

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