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A variant construction of Stiefel-Whitney classes of a topological manifold

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Topologia algebrica. — *A variant construction of Stiefel–Whitney classes of a topological manifold* (*). Nota di NICOLAE TELEMAN, presentata (**) dal Corrisp. E. MARTINELLI.

RIASSUNTO. — Sulla base del procedimento già introdotto in [3] per definire classi caratteristiche di fibrati con involuzione, costruisco qui le classi di Stiefel–Whitney delle varietà topologiche paracompatte con bordo. La costruzione ottenuta si presenta come una variante di quella di R. Thom.

§ 1. INTRODUCTION

It is known that for any paracompact topological manifold Stiefel-Whitney classes can be defined (see [1], [2]). In a recent paper [3] we defined a system of characteristic classes for fiber bundle with fiber-preserving involution, the fiber being an arbitrary $(n-1)$–simple space. Our construction has some common elements with the construction of the Steenrod squaring operations.

In this paper we use the technique developed in [3] to the case of topological manifolds X. For this aim, we consider a special class of coverings $\mathfrak{U}$ by "very small balls" U. The "tangent bundle of X" will be $\bigcup_{U \in \mathfrak{U}} U \times U$; the interchange of the coordinates in any product $U \times U$ induces an involution A in the tangent bundle, and the "non zero tangent vectors over U", $U \times U - \nabla_U$, is $(n-2)$–simple. Hence, the method developed in [3] can be applied.

There appears a difficulty; the involution A does not preserve the "fibers". This difficulty justifies the use of the "hereditary sequence of coverings" (HSC, § 2).

Certainly, this method permits us to construct Stiefel–Whitney characteristic classes, at the first step, for paracompact topological manifolds X without boundary. If X has the boundary $\dot{X}$, then $X \sqcup_{\dot{X}} X$ has not boundary and the restriction of the Stiefel–Whitney classes of $X \sqcup_{\dot{X}} X$ to X furnishes the Stiefel–Whitney classes of the manifold X.

In a successive Note, we propose to present an application of our construction to the problem of the topological invariance of the rational Pontrjagin classes.

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§ 2. HEREDITARY SEQUENCE OF COVERINGS

2.1. NOTATION. If (M, d) is an arbitrary metric space, let be $B_d(x_0, r) = \{x \mid x \in M, d(x, x_0) < r\}$.

2.2. DEFINITION. If X is a topologic manifold with boundary, $\dim X = n$, the open set $U \subset X$ is called an "open ball" in X if there exists an homeomorphism $f: B_d(x_0, 3) \to V$, where $B_d(x_0, 3) \subset \mathbf{E}^n_+$ ($\mathbf{E}^n_+ = \{(x^1, \cdots, x^n) \mid (x^1, \cdots, x^n) \in \mathbf{E}^n, x^1 \geq 0\}$, the Euclidean semispace) and $V \subset X$ is open, such that $f(B(x_0, 1)) = U$. We denote $f(B(x_0, r)) = rU$, $0 < r \leq 2$.

2.3. LEMMA. *If* X *is a paracompact topological manifold*, $\dim X = n$, *for any open covering* $\mathfrak{V} = \{V_\alpha\}_{\alpha \in \Lambda}$ *of* X *there exists an open covering* $\Sigma = \{\Sigma_a\}_{a \in A}$ *with the properties*:

(i) Σ_a, *for any* $a \in A$, *is an open ball in* X;
(ii) $2\Sigma = \{2\Sigma_a\}_{a \in A}$ *is a locally finite covering of* X, *finer that* $\mathfrak{V}$.

Proof. For any point $x \in V_\alpha$, $\alpha \in \Lambda$, we consider an open ball $V_{\alpha,x} \subset V_\alpha$ such that $2V_{\alpha,x} \subset V_\alpha$. Then, $\mathfrak{V}' = \{V_{\alpha,x}\}_{\alpha,x}$ is an open covering of X, finer than $\mathfrak{V}$. The manifold X being paracompact, there exist the open coverings $\mathfrak{W} = \{W_i\}_{i \in I}$, $\mathfrak{T} = \{T_i\}_{i \in I}$ such that:

1) $\bar{T}_i \subset W_i$;
2) $\mathfrak{W}$ is locally finite;
3) $\mathfrak{W}$ is finer than $\mathfrak{V}'$.

For any point $x \in \bar{T}_i$ we consider an open ball S_x which contains x such that $2S_x \subset W_i$. The space $\bar{T}_i$ being compact, and $\{S_x\}$ being an open covering of $\bar{T}_i$, there exists a finite subcovering $\mathfrak{S}_i = \{S_{x_1}, \cdots, S_{x_{p_i}}\}$. The covering $\Sigma = \bigcup_{i \in I} \mathfrak{S}_i$ has the desired properties.

2.4. DEFINITION. Let N be a fixed natural number, and T a topological space. The sequence $\mathfrak{U} = \{\mathfrak{U}^1, \mathfrak{U}^2, \cdots, \mathfrak{U}^n\}$ is called an "hereditary sequence of coverings of T, of length N and dimension n" (we shall write $\mathfrak{U} \in \mathrm{HSC}(T, N, n)$) if:

(i) for any $1 \leq i \leq N$, $\mathfrak{U}^i$ is an open covering of T;
(ii) any $U \in \mathfrak{U}^i$, $1 \leq i \leq N$, is an open n-ball in T;
(iii) for any $U, V \in \mathfrak{U}^i$, $1 \leq i \leq N-1$, $U \cap V \neq \varnothing$, there exists $W \in \mathfrak{U}^{i+1}$, such that $W \supset U \cup V$.

2.5. THEOREM. *If* X *is a paracompact topological manifold with boundary*, $\dim X = n$, *for any open covering* $\mathfrak{V}$ *of* X, *there exists* $\mathfrak{U} = \{\mathfrak{U}^1, \cdots, \mathfrak{U}^N\} \in \mathrm{HSC}(X, N, n)$ *with* $\mathfrak{U}^N$ *finer that* $\mathfrak{V}$.

Proof. We consider a covering $\Sigma = \{\Sigma_a\}_{a \in A}$ having the properties (i), (ii) from the Lemma 2.3. We know that for any $a \in A$ there exists an homeomorphism $f : B(x_a, 2) \to 2\Sigma_a$, $x_a \in \mathbf{E}^n_+$. We consider on every $2\Sigma_a$ the metric d_a defined by the formula:

$$d_a(x, y) = d(f^{-1}(x), f^{-1}(y)),$$

d being the Euclidean metric.

Let $a, b \in A$ be two arbitrary indices, such that $3/2\,\Sigma_a \cap 3/2\,\Sigma_b \neq \varnothing$. On $C_{ab} = \overline{3/2\,\Sigma_a} \cap \overline{3/2\,\Sigma_b}$, we have two metrics: d_a and d_b. We affirm that for any $\varepsilon_b > 0$ there exists $\varepsilon_a > 0$ such that

$$d_a(x, y) < \varepsilon_a \Rightarrow d_b(x, y) < \varepsilon_b, \qquad x, y \in C_{ab}.$$

Really, on $C_{ab} \times C_{ab}$ let be the product metric δ_a:

$$\delta_a^2((x, y), (x_0, y_0)) = d_a^2(x, x_0) + d_a^2(y, y_0).$$

The function

$$d_b : C_{ab} \times C_{ab} \to \mathbf{R}$$

being continuous on the compact space $C_{ab} \times C_{ab}$, is uniformly continuous, i.e., for any $\varepsilon_b > 0$ there exists $\varepsilon_a > 0$ such that

$$\delta_a((x, y), (x_0, y_0)) < \varepsilon_a \Rightarrow |d_b(x, y) - d_b(x_0, y_0)| < \varepsilon_b,$$

for any $(x, y), (x_0, y_0) \in C_{ab} \times C_{ab}$.

In particular, if we take $y = x_0 = y_0$, we obtain

$$\delta_a((x, y), (x_0, y_0)) = d_a(x, y) < \varepsilon_a \Rightarrow d_b(x, y) < \varepsilon_b.$$

Let be $A_a = \{b \mid b \in A, \overline{3/2\,\Sigma_a} \cap \overline{3/2\,\Sigma_b} \neq \varnothing\}$. Let be ε_{ab} the corresponding ε_a for $\varepsilon_b = 1/6$, $b \in A_a$. Let

$$\delta_{ab} = \inf\{d_b(x, y) \mid x \in (\partial\Sigma_a) \cap \Sigma_b, \; y \in \partial(3/2\,\Sigma_a) \cap \Sigma_b, \; b \in A_a\};$$

we have $\delta_{ab} > 0$, and if $x \in \Sigma_a \cap \Sigma_b$, $y \in \Sigma_b$ such that $d_b(x, y) < \delta_{ab}$, then $y \in 3/2\,\Sigma_a$. Really, let l be the segment (by respect the metric d_b) which connects the points x, y. If $l \cap \partial\Sigma_a = \varnothing$, $l \subset \Sigma_a$ and the assertion is proved; in the contrary case, let be ξ an arbitrary point of $l \cap \partial\Sigma_a$. If $l \cap \partial(3/2\,\Sigma_a) = \varnothing$, the assertion is proved again. We suppose the absurd, i.e., there exists $\eta \in l \cap \partial(3/2\,\Sigma_a)$; then we have

$$\delta_{ab} > d_b(x, y) > d_b(\xi, \eta) \geq \delta_{ab}$$

which is impossible.

Let be $r_a = \min\limits_{b \in A_a}\{\varepsilon_{ab}, 1/2 \cdot \delta_{ab}, 1/6\}$.

For any point $x \in \Sigma_a$, we consider a ball

$$B_{d_a}(x, r_x) \subset \Sigma_a, \qquad r_x < r_a.$$

Let be $\mathfrak{A}^N = \{B_{d_a}(x, r_x)\}_{a \in A, x \in \Sigma_a}$.

The covering $\mathfrak{A}^N$ has the properties:

(1) $\mathfrak{A}^N$ is an open covering of X;

(2) any $U \in \mathfrak{A}^N$ is a n–ball in X;

(3) if $U, V \in \mathfrak{A}^N$, $U \cap V \neq \varnothing$, then there exists $W \in \mathfrak{V}$ such that $U \cup V \subset W$.

Only (3) needs a proof. Let be $U = B_{d_a}(x, r_x)$, $V = B_{d_b}(y, r_y)$, $a, b \in A$, and let be $\xi \in U \cap V$. We have for any $z \in B_{d_b}(y, r_y) : d_b(\xi, z) \leq \leq 2 r_b \leq 2 \cdot \frac{1}{2} \delta_{ab} = \delta_{ab}$, hence $B_{d_b}(y, r_y) \subset 3/2\, \Sigma_a$, and $d_a(\xi, z) \leq d_a(\xi, y) + + d_a(z, y) < 1/6 + 1/6 = 1/3$ because $d_b(\xi, y) < r_b \leq \varepsilon_{ba}$, $d_b(z, y) < r_b \leq \varepsilon_{ba}$. We have in consequence $B_{d_b}(y, r_y) \subset B_{d_a}(\xi, 1/3)$; also $B_{d_a}(x, r_x) \subset B_{d_a}(\xi, 1/3)$ and $B_{d_a}(\xi, 1/3) \subset 3/2\, \Sigma_a \subset V_{f(a)}$.

If we take in (3) $U = V$, we deduce the covering $\mathfrak{A}^N$ is finer that $\mathfrak{V}$.

If we repeat the upper construction changing $\mathfrak{V}$ by $\mathfrak{A}^N$, we obtain a new covering which we denote by $\mathfrak{A}^{N-1}$, and so on.

2.6. COROLLARY. HSC (X, N, n) *contains at least one element for arbitrary* N.

§ 3. STIEFEL–WHITNEY CLASSES OF PARACOMPACT TOPOLOGICAL MANIFOLDS

3.1. DEFINITION. Let X be a topological manifold without boundary, $\dim X = n$. Let $\mathfrak{A} = \{U_\alpha\}_{\alpha \in \Lambda}$ be an open covering such that any U_α is a ball.

The "$\mathfrak{A}$–tangent bundle of X", $T^{\mathfrak{A}} X$, is

$$T^{\mathfrak{A}} X = \bigcup_{\alpha \in \Lambda} U_\alpha \times U_\alpha \subset X \times X .$$

The tangent bundle admits a natural open covering $\mathfrak{A}^* = \{U_\alpha \times U_\alpha\}_{\alpha \in \Lambda}$. The points of $T^{\mathfrak{A}} X$ are called "tangent vectors" and the points $(x, x) \in T^{\mathfrak{A}} X$ are called "zero tangent vectors"; let be $T_0 X = T^{\mathfrak{A}} X - \nabla_X$, where $\nabla_X = \{(x, x) \mid x \in X\}$ is the set of zero tangent vectors of X. Let $\mathfrak{A}_0^*$ denote the covering of $T_0^{\mathfrak{A}} X$ induced by the covering $\mathfrak{A}^*$ on the subspace $T_0^{\mathfrak{A}} X \subset T^{\mathfrak{A}} X$.

3.2. LEMMA. $U_\alpha \times U_\alpha - \nabla_{U_\alpha}$ *is homeomorphic to* $\mathbf{R}_0^n \times \mathbf{R}^n$ (see [2], Chap. 6,2.5).

Proof. We can take $U = \mathbf{R}^n$; the homeomorphism $\mathbf{R}^n \times \mathbf{R}^n \xrightarrow{h} \mathbf{R}^n \times \mathbf{R}^n$: $(x, y) \overset{h}{\longmapsto} (x - y, x)$ proves the assertion.

3.3. NOTATION. If M is a topological space and $\mathfrak{A}$ is a covering of M, let $C_*^{\mathfrak{A}}(M, R)$ denote the subcomplex of the singular chain complex $C_*(M, R)$ which is generated by the singular simplexes having the image in at least one of the sets of the covering $\mathfrak{A}$ (R being an arbitrary commutative ring with 1). Let be $C_{\mathfrak{A}}^*(M, R) = \hom(C_*^{\mathfrak{A}}(M, \mathbf{Z}), R)$.

3.4. DEFINITION. Let $A : (T^{\mathfrak{U}} X, T_0^{\mathfrak{U}} X) \hookleftarrow$ be the continuous involution $A : (x_0, x_1) \mapsto (x_1, x_0)$.

Let X be a paracompact topological manifold, $\dim X = n$, and $Y = X \times \mathbf{R}$. If $\mathfrak{U} = \{V_\alpha\}_{\alpha \in \Lambda}$ is a covering of X, let be $\hat{V}_\alpha = V_\alpha \times \mathbf{R}$, and let $\hat{\mathfrak{V}}$ denote the covering $\{\hat{V}_\alpha\}_{\alpha \in \Lambda}$ of Y; if

$$\mathfrak{U} = \{\mathfrak{U}^1, \cdots, \mathfrak{U}^N\} \in \mathrm{HSC}(X, N, n), \qquad \text{then}$$

$$\hat{\mathfrak{U}} = \{\hat{\mathfrak{U}}^1, \cdots, \hat{\mathfrak{U}}^N\} \in \mathrm{HSC}(Y, N, n + 1).$$

Let $s : Y \to Y \times Y$ be the map

$$s : (x, t) \mapsto ((x, t - 1), (x, t + 1)), \qquad (x, t) \in X \times \mathbf{R}.$$

3.5. CONVENTION. If $\mathfrak{U} \in \mathrm{HSC}(X, N, n)$, let us denote $\hat{\mathfrak{U}}$, $\bar{\hat{\mathfrak{U}}}$ and $\bar{\hat{\mathfrak{U}}}_0$ also by $\mathfrak{U}$ if it does not produce any confusion.

3.6. PROPERTY. For any open covering $\mathfrak{U}$ of X, $s(Y) \subset T_0^{\mathfrak{U}} Y$.

If $\mathfrak{V}$ is an open covering of X, finer that $\mathfrak{U}$, then the chain map $\bar{s}_*$ induced by s has the property:

$$\bar{s}_* (C_*^{\mathfrak{V}}(Y, R)) \subset C_*^{\mathfrak{U}} (T_0^{\mathfrak{U}} Y, R).$$

3.7. THEOREM. *If* X *is a paracompact topological manifold without boundary*, $\dim X = n$, *and* $Y = X \times \mathbf{R}$, *then there exists a natural number* $\nu(n)$ *such that for any* $\mathfrak{U} = (\mathfrak{U}^1, \cdots, \mathfrak{U}^{2\nu(n)}) \in \mathrm{HSC}(X, 2\nu(n), n)$ *and for any open covering* $\mathfrak{V} = \{V_\alpha\}_{\alpha \in \Lambda}$ *finer than* $\mathfrak{U}^1$, *there exists the "pseudo–local"* R*–homomorphisms*:

$$k_p^{(r)} : C_p^{\mathfrak{V}}(Y, R) \to C_{p+r}^{\mathfrak{V}^{\nu(n)}} (T_0^{\mathfrak{U}^{\nu(n)}} Y, R), \qquad p + r \le n,$$

(i) $\quad k_p^{(0)} = \bar{s}_p$

$$(1 + (-1)^r A)\, k_p^{(r-1)} = \partial k_p^{(r)} + (-1)^{r-1} k_{p-1}^{(r)} \partial_p,$$

(*"pseudo-local" means: for any singular simplex* $\sigma \in C_p^{\mathfrak{V}}(Y, R)$ *let* $V(\sigma) \in \mathfrak{V}$ *be a fixed set of the covering* $\mathfrak{V}$ *such that* $\sigma \in C_p(V(\sigma) \times \mathbf{R}, R)$; *then there exists* $U \in \mathfrak{U}^{\nu(n)}$, $V(\sigma) \subset U$, *such that*

$$k_p^{(r)}(\sigma) \in C_{p+r}(\hat{U} \times \hat{U} - \nabla_{\hat{U}}, R)).$$

If $k_p^{(r)}$, $\tilde{k}_p^{(r)}$ *are two such systems of homomorphisms which satisfy* (i), *then there exist the pseudo-local* R*–homomorphisms*:

$$\varphi_p^{(r)} : C_p^{\mathfrak{V}}(Y, R) \to C_{p+r+1}^{\mathfrak{U}^{2\nu(n)}} (T_0^{\mathfrak{U}^{2\nu(n)}} Y, R), \qquad p + r + 1 \le n,$$

(ii) $\quad$ *such that, if we denote* $K_p^{(r)} = k_p^{(r)} - \tilde{k}_p^{(r)}$, *we have*

$$K_p^{(r)} = (1 + (-1)^r A)\, \varphi_p^{(r-1)} + \partial \varphi_p^{(r)} + (-1)^r \varphi_{p-1}^{(r)} \partial,$$

and the pseudo-local R*–homomorphisms*:

$$\mu_{n-r}^{(r)} : C_{n-r}^{\mathfrak{V}} (Y , R) \to C_n^{\mathfrak{A}^{2\nu(n)}} (T_0^{\mathfrak{A}^{2\nu(n)}} Y , R)$$

such that

$$\text{(iii)} \qquad K_{n-r}^{(r)} = (1+(-1)^r A)\, \varphi_{n-r}^{(r-1)} + (-1)^r \varphi_{n-r-1}^{(r)} \partial + \mu_{n-r}^{(r)} \qquad \partial\mu_{n-r}^{(r)} = 0 .$$

Proof. The proof of the Theorem is essentially the proof of the Theorem 7.1 [3]. We shall indicate here only the specific part of the proof.

Let $\sigma_0 \in C_0^{\mathfrak{V}} (Y , R)$ be a singular simplex; this simplex belongs to $C_0 (V(\sigma_0) , R)$. There exists at least one $U(\sigma_0) \in \mathfrak{A}^1$ such that $V(\sigma_0) \subset U(\sigma_0)$. In $C_* (\widehat{U(\sigma_0)} \times \widehat{U(\sigma_0)} - \nabla_{\widehat{U(\sigma_0)}} , R)$ we can solve the equation (see the Lemma 3.2)

$$\partial (k_0^{(1)} (\sigma_0)) = (1 - A)\, s_0 (\sigma_0) .$$

Let be σ_1 a singular simplex of $C_1 (V(\sigma_1) , R) \subset C_1^{\mathfrak{V}} (Y , R)$, $\partial\sigma_1 = x_1 - x_0$. We affirm that there exists $U(\sigma_1) \in \mathfrak{A}^3$ such that

$$\gamma = (1 - A)\, s_1 (\sigma_1) - k_0^{(1)} \partial\sigma_1 \in C_1 (U(\sigma_1) \times U(\sigma_1) - \nabla_{U(\sigma_1)} , R) ;$$

really, there exists

$$U_1 \in \mathfrak{A}^1 \quad \text{such that} \quad V(\sigma_1) \subset U_1 \quad \text{and}$$

$$k_0^{(1)} x_i \in C_1 (U (x_i) \times U (x_1) - \nabla_{U(x_i)} , R), \quad U (x_i) \in \mathfrak{A}^1, \qquad i = 0 , 1 .$$

But $U_1 \cap U(x_1) \neq \varnothing \neq U_1 \cap U(x_0)$; hence, there exists $U', U'' \in \mathfrak{A}^2$ such that

$$U_1 \cup U(x_1) \subset U' \quad , \quad U_1 \cup U(x_0) \subset U'' ;$$

because $U' \cap U'' \neq 0$, there exists $U (\sigma_1) \in \mathfrak{A}^3$ such that $U' \cup U'' \subset U (\sigma_1)$.

Now we can solve the equation $\partial k_p^{(1)} (\sigma_1) = \gamma$ because γ is a cycle in C_* $(U(\sigma_1) \times U(\sigma_1) - \nabla_{U(\sigma_1)} , R)$, when n is sufficiently great. We call $U(\sigma_0)$, $U(\sigma_1)$ the supports of $k_0^1 (\sigma_0)$, $k_1^{(1)} (\sigma_1)$.

We continue this construction by the increased induction over p and r. We observe that for any $k_p^{(r)}$ (if all $k_{p'}^{(r')}$, $r' \leq r$, $p' < p$, are constructed) we must increase the supports a finite number of times; therefore, for all $k_p^{(r)}$, $p + r \leq n$, we must perform only a finite number, $\nu' (n)$, of augmentations of the supports. Also, for the construction of $\varphi_p^{(r)}$, $\mu_{n-r}^{(r)}$, $p + r + 1 \leq n$, we must perform a finite number, $\nu''(n)$, of similar constructions. Let $\nu (n) = \max (\nu'(n) , \nu''(n))$.

3.8. DEFINITION. Let $X , Y , \mathfrak{V} , \mathfrak{A} , k_p^{(r)}$ be as in the Theorem 3.7. We define:

$$\omega_i (k_p^{(r)}) \in C^{n-i+1} (Y , \mathbf{Z}_2) , \qquad 0 \leq i \leq n ,$$

$$(\omega_i (k_p^{(r)})) (\sigma) = [\{(1 + A)\, k_i^{(n-i)} + k_{i-1}^{(n-i+1)} \partial_i\} (\sigma)]$$

where $[\gamma]$ denotes the homology class of the cycle γ in $H_n(U \times U - \nabla_U, \mathbf{Z}_2) \simeq \mathbf{Z}_2$, $U \in \mathfrak{U}^{\nu(n)}$ being a support of $\gamma = \{(1 + A)\, k_i^{(n-i)} + k_{i-1}^{(n-i+1)}\, \partial_i\}\,(\sigma)$ (the homology class of γ is independent of the support U; see [2] Chap. 6,2.5).

3.9. THEOREM. $d\omega_i(k_p^{(r)}) = 0, \qquad 0 \leq i \leq n.$

Proof. The assertion of the Theorem follows as follows the Theorem 7.4. [3], it being an algebraic consequence of the relations (i) Theorem 3.7.

3.10. THEOREM. *The cohomology class of* $\omega_i(k_p^{(r)})$, $0 \leq i \leq n$, *is independent of the choice of* $k_p^{(r)}$.

Proof. See the proof of the Theorem 7.5. [3].

3.11. DEFINITION. Let be $\overline{W}_i(Y, \mathfrak{U}, \mathfrak{V}) \in H^i_{\mathfrak{V}}(Y, \mathbf{Z}_2)$, $0 \leq i \leq n$, the cohomology class of the cocycle $\omega_i(k_p^{(r)})$. Let $W_i(Y, \mathfrak{U}, \mathfrak{V}) \in H^i(Y, \mathbf{Z}_2)$ be the unique cohomology class which corresponds to $\overline{W}^i(X, \mathfrak{U}, \mathfrak{V})$ by the inclusion $j : C_*^{\mathfrak{V}}(Y, \mathbf{Z}_2) \to C_*(Y, \mathbf{Z}_2)$ which is a chain equivalence.

3.12. THEOREM. $W_i(Y, \mathfrak{U}, \mathfrak{V})$, $0 \leq i \leq n$, *does not depend on the choice of* $\mathfrak{U}$ *and* $\mathfrak{V}$ ($\mathfrak{U} \in \mathrm{HSC}(Y, 2\nu(n), n)$, $\mathfrak{V}$ *an open covering of* X, $\mathfrak{V}$ *finer than* $\mathfrak{U}^1$).

Proof. Let $\mathfrak{V}_i$, $i = 1, 2$ be two coverings of X, $\mathfrak{U}_i \in \mathrm{HSC}(X, 2\nu(n), n)$, $\mathfrak{V}_i$ finer than $\mathfrak{U}_i$. Let be $\mathfrak{U} = \{\mathfrak{U}^1, \cdots, \mathfrak{U}^{2\nu(n)}\} \in \mathrm{HSC}(X, 2\nu(n), n)$ such that $\mathfrak{U}^{2\nu(n)}$ be finer than $\mathfrak{V}_1 \cap \mathfrak{V}_2$ (the existence of $\mathfrak{U}$ is assured by the Theorem 2.5). Let be

$$k_p^{(r)} : C_p^{\mathfrak{U}'}(Y, \mathbf{Z}_2) \to C_{p+r}^{\mathfrak{U}^{\nu(n)}}(T_0^{\mathfrak{U}^{\nu(n)}} Y, \mathbf{Z}_2), \qquad p + r \leq n$$

an arbitrary system of pseudo-local homomorphisms as in (i) Theorem 3.7. In the sequence

$$\mathfrak{U}^{2\nu(n)}, \mathfrak{V}_1 \cap \mathfrak{V}_2, \mathfrak{V}_i, \mathfrak{U}_i^1, \mathfrak{U}_i^{\nu(n)}, \qquad i = 1, 2,$$

any covering is finer than the successive; therefore, we can consider

$$k_p^{\prime(r)} : C_p^{\mathfrak{V}_1}(Y, \mathbf{Z}_2) \to C_{p+r}^{\mathfrak{U}_1^{\nu(n)}}(T_0^{\mathfrak{U}_1^{\nu(n)}} Y, \mathbf{Z}_2), \qquad \text{resp.}$$

$$k_p^{\prime\prime(r)} : C_p^{\mathfrak{V}_2}(Y, \mathbf{Z}_2) \to C_{p+r}^{\mathfrak{U}_1^{\nu(n)}}(T_0^{\mathfrak{U}_2^{\nu(n)}} Y, \mathbf{Z}_2)$$

defined only on the singular simplexes of $C_p^{\mathfrak{V}_1}(Y, \mathbf{Z}_2)$, resp. $C_p^{\mathfrak{V}_2}(Y, \mathbf{Z}_2)$ which lie in $C_p^{\mathfrak{U}^1}(Y, \mathbf{Z}_2)$. The bide systems of pseudo-local homomorphisms can be extended over all $C_p^{\mathfrak{V}_1}(Y, \mathbf{Z}_2)$, resp. $C_p^{\mathfrak{V}_2}(Y, \mathbf{Z}_2)$; we denote them by $k_p^{\prime(r)}$, resp. $k_p^{\prime\prime(r)}$. When we consider the corresponding cocyles $\omega_i(k_p^{(r)})$, $\omega_i(k_p^{\prime(r)})$, $\omega_i(k_p^{\prime\prime(r)})$, we have, by construction:

$$\omega_i(k_p^{\prime(r)})\Big|_{C_i^{\mathfrak{U}'}(Y, \mathbf{Z}_2)} = \omega_i(k_p^{\prime\prime(r)})\Big|_{C_i^{\mathfrak{U}'}(Y, \mathbf{Z}_2)} = \omega_i(k_p^{(r)});$$

the inclusions of complexes

$$C_*^{\mathfrak{U}^1}(Y, \mathbf{Z}_2) \begin{array}{l} \xrightarrow{j_1} C_*^{\mathfrak{V}_1}(Y, \mathbf{Z}_2) \\ \xrightarrow{j_2} C_*^{\mathfrak{V}_2}(Y, \mathbf{Z}_2) \end{array}$$

being chain equivalences, the theorem follows.

3.13. DEFINITION. The classes $\mathfrak{W}_i(Y, \mathfrak{U}, \mathfrak{V})$ define unique classes $\mathfrak{W}_i(X) \in H^i(X, \mathbf{Z}_2) \simeq H^i(Y, \mathbf{Z}_2)$ which are called the "Stiefel–Whitney classes of the paracompact topological manifold without boundary X".

If X is a paracompact topological manifold with boundary $\dot{X}$, let be $\widehat{X} = X \sqcup_{\dot{X}} X$; $\widehat{X}$ is a paracompact topological manifold without boundary, and let $j : X \hookrightarrow \widehat{X}$ be the inclusion on one of the summands.

3.14. DEFINITION. The Stiefel–Whitney classes $\mathfrak{W}_i(X)$, $0 \leq i \leq n$, of the paracompact topological manifold with boundary X, $\dim X = n$, are

$$\mathfrak{W}_i(X) = j^* \mathfrak{W}_i(\widehat{X}).$$

3.15. THEOREM. *If* X *is a paracompact topological manifold which admits a differential structure, let* T (X) *denote the corresponding tangent bundle. Then*

$$\mathfrak{W}_i(X) = W_i(T(X)).$$

Proof. Let h be a Riemannian metric on X, and let $\bar{h}$ denote the product metric on $Y = X \times \mathbf{R}$, where $\mathbf{R}$ has the canonic Euclidean metric. Let d be the associated metric on Y. For any point $x_0 \in Y$ there exists a positive number r_{x_0} such that for any two points $x, y, d(x, x_0), d(y, y_0) < r_{x_0}$, there exists a unique "small" geodesic $\overline{xy}$ which connects them; let $m \in \overline{xy}$ be his half. We identify the small tangent vectors with their image in Y by the exponential map. To the pair $(x, y) \in B_d(x_0, r_{x_0})$ we associate the point $x \in T_m Y$. In this manner, we define an homeomorphism:

$$\rho : T^{\mathfrak{U}} Y \to Y \quad , \quad V \subset TX,$$

$$\mathfrak{U} \text{ being the covering } \quad \mathfrak{U} = \{B_d(x_0, r_{x_0})\}_{x_0 \in Y} \quad \text{ of } Y$$

and V an open neighbourhood of the zero section in TX. We remark that the involution A in $T^{\mathfrak{U}} Y$ corresponds by ρ to the *antipodal* map in $V \subset TX$. Now the theorem follows from the Theorem 9.4, [3] and the definitions of $\mathfrak{W}_i(X)$, and $\mathfrak{W}_i(T(X))$.

Note. In a further note we shall study the connection between the Stiefel-Whitney classes defined by Thom and the classes $W_i(X)$ when the manifold X has no differentiable structure.

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