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RIASSUNTO. — Vedi la Nota IV, apparsa alle pp. 402–409, vol. LIII di questi «Rendiconti».

§ 5. CUBICAL CARRIERS. CATEGORY CH.

We will state and prove the cubical carrier theorem based on (4.8). Let $G = \{g_t : (V, V_0) \to (W, W_0) ; t \in T\}$ be a family of cubical morphisms admitting a common carrier (see [2], Definition 1.1). We define the acyclic cubical carrier $E(G)$ of G as follows:

(5.1) Let F_β be a face of V.

$$E(G)(F_\beta) \overset{\text{df}}{=} C(\operatorname{carr}(\bigcup_t f_{q_t}(F_\beta))).$$

[Note that by (4.8) $\operatorname{carr}(\bigcup_t f_{q_t}(F_\beta))$ is acyclic and that $E(G)(F_\beta) \subset C(W_0)$ for $F_\beta \subset V_0$].

(5.2) DEFINITION. A chain homomorphism $h : C(V) \to C(W)$ is said to be carried by $E(G)$ iff $h(F_\beta) \in E(G)(F_\beta)$ for each $F_\beta \subset V$.
[Note that $h(C(V_0)) \subset C(W_0)$].

(5.3) THEOREM. *Let $G = \{g_t : (V, V_0) \to (W, W_0) ; t \in T\}$ be a family of cubical morphisms admitting a common cubical carrier. Then there exists a chain homomorphism $h : C(V) \to C(W)$ carried by* $E(G)$.

Proof. We define $h_{-1} = \mathrm{Id}_Z : Z \to Z$. The proof proceeds by induction. Suppose that h has been extended over all of $C_k(V)$. Assume that $F_\beta \in C_{k+1}(V)$. $\gamma = h(\delta F_\beta)$ is a cycle in $C_k(E(G))(F_\beta)$. Thus, there exists $\tilde{\gamma} \in C_{k+1}(E(G))(F_\beta)$ such that $\delta\tilde{\gamma} = \gamma$. We define $h(F_\beta) = \tilde{\gamma}$.

(5.4) THEOREM. *The map h given by the theorem* (5.3) *is unique up to a chain homotopy carried by* $E(G)$.

Suppose that h^1 and h^2 are two chain homomorphisms carried by $E(G)$. Let $h = h^1 - h^2$. We will show that h is chain homotopic to zero. Note that

$$h_{-1} = h^1_{-1} - h^2_{-1} = 0 .$$

(*) Nella seduta del 9 dicembre 1972.

We will inductively construct a chain homotopy Δ carried by $E(G)$. We define $\Delta_0 = 0$. Suppose that Δ has been extended over all of $C_k(V)$ and that Δ is carried by $E(G)$. Let $F_\beta \in C_{k+1}(V)$.

$$\begin{aligned} &\gamma = (\Delta\delta)(F_\beta) - h(F_\beta) \quad \text{is a cycle in} \\ &C_{k+1}(E(G)(F_\beta)), \quad \text{for:} \\ &\delta\gamma = \delta\Delta\delta(F_\beta) - \delta h(F_\beta) = (h - \Delta\delta)\,\delta(F_\beta) - \delta h(F_\beta) = \\ &\qquad = h\delta(F_\beta) - \delta h(F_\beta) = 0 . \end{aligned}$$

Hence there exists $\tilde{\gamma} \in C_{k+2}(E(G)(F_\beta))$ such that $\delta\tilde{\gamma} = -\gamma$. We define $\Delta(F_\beta) = \tilde{\gamma}$. This concludes our construction.

As an immediate corollary we have the following:

(5.5) THEOREM. *Let* $q : A_1 \to A$ *be a cubical morphism of* $(V, V_0) \to (W, W_0)$. *Then there exists a chain map* $h : C(V, V_0) \to C(W, W_0)$ *such that* $h(F_\beta) \subset C(\mathit{carr}\, f_q(F_\beta))$ *and h is uniquely determined up to chain homotopy.*

Proof. Apply (5.3) and (5.4) to $G = \{q\}$.

We will denote the chain homotopy class of h by $C(q)$.

(5.6) COROLLARY. *If* $G = \{g_t : (V, V_0) \to (W, W_0)\}$ *is a family of cubical morphisms admitting a common carrier, then* $C(q_{t_1}) = C(q_{t_2})$ *for every* $t_1, t_2 \in T$.

Proof. Apply (5.4) to $C(q_{t_1})$ and $C(q_{t_2})$.

(5.7) COROLLARY. *If cubical morphisms* $p, q : (V, V_0) \to (W, W_0)$ *are contiguous then* $C(p) = C(q)$.

(5.8) COROLLARY. *Let* $q : (V, V_0) \to (W, W_0)$ *and* $q_1 : (W, W_0) \to (U, U_0)$ *be cubical morphisms. Then*:

a) $C(q_1 \circ q) = C(q_1) \circ C(q)$

b) *if* $q = \mathrm{Id}_A : A \to A$ *then* $C(q) : (V, V_0) \to (V, V_0) = \mathrm{Id}_{C(V, V_0)}$.

Let Ch denote the category of chain complexes and chain homotopy classes of chain maps. Thus we have obtained a functor C from the category QP_0 of finite cubical pairs into Ch:

(5.9)
$$\begin{gathered} (V, V_0) \mapsto C(V, V_0) \\ [q : (V, V_0) \to (W, W_0)] \mapsto C(q) . \end{gathered}$$

By (5.7) there is a factorization of the functor C:

(5.10)
$$C = \tilde{C} \circ Cg .$$

The functor $\tilde{C} : CQ_0 \to \mathrm{Ch}$ is unique.

[CQ_0 denotes here contiguous category restricted to finite polyhedra].

(5.10) *Remark.* We will show that up to isomorphism $C(V, V_0)$ does not depend on the orientation. Let or′ be another orientation of the cube I^A. We define a natural transformation

$$\Gamma : C(V, V_0\,;\ \mathrm{or}) \rightarrow C(V, V_0\,;\ \mathrm{or}')$$

as follows:

$$\Gamma([F_\beta\,,\ \mathrm{or}\,F_\beta]) = [\mathrm{or}\,F_\beta : \mathrm{or}' \cdot F_\beta] \cdot [F_\beta\,,\ \mathrm{or}'\,F_\beta]\,.$$

It is easy to notice that

$$\Gamma' : C(V\,, V_0\,;\ \mathrm{or}') \rightarrow C(V\,, V_0\,;\ \mathrm{or})$$

given by

$$\Gamma([F_\beta\,,\ \mathrm{or}'\,F_\beta]) = [\mathrm{or}'\,F_\beta : \mathrm{or}\,F_\beta] \cdot [F_\beta\,,\ \mathrm{or}\,F_\beta]$$

is the inverse of Γ. Thus up to the natural equivalence C does not depend on the orientation.

§ 6. Combinatorial homology of cubical polyhedra

We are now in a position to define homology for cubical polyhedra.

(6.1) Definition. Let G be a group.

$$H_k(V, V_0\,; G) \overset{\mathrm{df}}{=} H_k(C(V, V_0)\,; G)$$
$$H^k(V, V_0\,; G) \overset{\mathrm{df}}{=} H^k(C(V, V_0)\,; G)\,.$$

$H_k(V, V_0)$ will be used to denote $H_k(V, V_0\,; Z)$.

We will now show that our homology satisfies a set of axioms that may be considered as the combinatorial equivalent of the Eilenberg-Steenrod's axioms for homology.

(6.2) Lemma. *Let* $V_0 \subset V \subset I^A$ *be a pair of cubical polyhedra. Let* $i : C(V_0) \rightarrow C(V)$ *be the canonical injection. Then*

$$C(\mathrm{Id}_A : V_0 \rightarrow V) = [i]\,.$$

Proof. Put $G = \{\mathrm{Id}_A : V_0 \rightarrow V\}$. i is carried by $E(G)$. Hence by (5.4) $[i] = C(\mathrm{Id}_A)$.

(6.3) Lemma. *Let* $V_0 \subset V \subset I^A$ *be a pair of cubical polyhedra. Let* $j : C(V) \rightarrow C(V, V_0)$ *be the canonical projection. Then* $C(\mathrm{Id}_A : (V, \varnothing) \rightarrow (V, V_0)) = [j]$.

Thus the algebraic homomorphisms i and j are induced by the respective cubical morphisms. From the definition of $C(V, V_0)$ we have an exact sequence

$$0 \rightarrow C(V_0) \xrightarrow{i} C(V) \xrightarrow{j} C(V, V_0) \rightarrow 0 \qquad (6.4)$$

for every pair of cubical polyhedra (V, V_0).

From (6.2) and (6.3) we have $H([i]) = H(C(\mathrm{Id}_A : V_0 \to V))$ and $H([j]) = H(\mathrm{Id}_A : (V, \varnothing) \to (V, V_0))$. It is a standard conclusion in homological algebra that (6.4) gives us an exact sequence:

$$\to H_k(V_0) \to H_k(V) \to H_k(V, V_0) \xrightarrow{\delta} H_{k-1}(V_0) \to \cdots$$

We now establish an excision axiom. Let $(V, V_0) \subset I^A$ be a pair of cubical polyhedra. Notice that $V \setminus \mathrm{Int}_V V_0 \subset I^A$ is a polyhedron (if $x \in V \setminus \mathrm{Int}_V V_0$ then $\mathrm{carr}(x)$ is a face in $V \setminus \mathrm{Int}_V V_0$). By considering generators, we obtain the canonical chain isomorphism

$$C(V \setminus \mathrm{Int}_V V_0 ; V_0 \setminus \mathrm{Int}_V V_0) \simeq C(V, V_0) .$$

Hence, we have

(6.5) If (V, V_0) is a polyhedral pair in I^A then

$$H(V \setminus \mathrm{Int}_V V_0 ; V_0 \setminus \mathrm{Int}_V V_0) \simeq H_k(V, V_0) ,$$

and this isomorphism is induced by the inclusion

$$(V \setminus \mathrm{Int}_V V_0 , V_0 \setminus \mathrm{Int}_V V_0) \to (V, V_0) .$$

§ 7. EILENBERG-STEENROD AXIOMS FOR FINITE CUBICAL COMPLEXES

There is a sequence of functors $\{H_k\}_{k=0}^{\infty}$ from the category of finite cubical pairs into the category of groups and a function $\delta = \delta(k, V, V_0)$: $H_k(V, V_0) : H_k(V, V_0) \to H_{k-1}(V_0)$ such that

1) If $g : A_1 \to A$ is a cubical morphism of (V, V_0) into (W, W_0) then the following diagram commutes:

$$\begin{array}{ccc} H_k(V,V_0) & \xrightarrow{H_k(g)} & H_k(W,W_0) \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H_{k-1}(V_0) & \xrightarrow{H_{k-1}(g)} & H_{k-1}(W_0) \end{array}$$

2) *Strong excision axiom.*

If $(V, V_0) \subset I^A$ is a cubical pair and $\mathrm{Id}_A : A \to A$ is an inclusion morphism of $(V \setminus \mathrm{Int}_V V_0 , V_0 \setminus \mathrm{Int}_V V_0) \to (V, V_0)$, then

$$H(\mathrm{Id}_A) : H(V \setminus \mathrm{Int}_V V_0 , V_0 \setminus \mathrm{Int}_V V_0) \to H(V, V_0)$$

is an isomorphism.

3) *Exactness axiom.*

If $(V, V_0) \subset I^A$ is a cubical pair and $\mathrm{Id}_A : A \to A$ is a morphism of V_0 into V, and of V into (V, V_0) then the sequence

$$\cdots \to H_k(V_0) \xrightarrow{H_k(\mathrm{Id}_A)} H_k(V) \xrightarrow{H_k(\mathrm{Id}_A)} H_k(V, V_0) \xrightarrow{\delta} H_{k-1}(V_0) \to$$

is exact.

4) *Homotopy axiom.*

If two cubical morphisms $p, q : (V, V_0) \to (W, W_0)$ are contiguous then $H(p) = H(q) : H(V, V_0) \to H(W, W_0)$.

5) *Dimension axiom.*

If $P = F_\beta \subset I^A$ is a vertex of I^A then

$$H_k(P) = 0 \qquad \text{for} \quad k \neq 0 .$$

§ 8. Cubical homology of compact pairs

Using results and definitions of preceeding sections we will define the homology of a pair of compact topological spaces. Throughout this chapter all spaces will be compact. Let us recall ([1], Property 3.8) that

$$Q(X, X_0) = \varprojlim (N^A_B(X, X_0), i^A_B : B \subset A \in \mathrm{Fin}\, A(X)) .$$

We define the cubical homology of a compact pair.

(8.1) Definition. Let G be a group.

$$H_k(X, X_0; G) \overset{\mathrm{df}}{=} \varprojlim (H_k(N^A_B(X, X_0); G), H_k(i^A_B) : B \subset A \in \mathrm{Fin}\, A(X))$$

$$H^k(X, X_0; G) \overset{\mathrm{df}}{=} \varinjlim (H^k(N^A_B(X, X_0); G), H^k(i^A_B) : B \subset A \in \mathrm{Fin}\, A(X)).$$

$H_k(X, X_0)$ will be used to denote $H_k(V, V_0; Z)$.

Let $g : (X, X_0) \to (Y, Y_0)$ be a continuous map. Let us recall ([1], 3.9) that:

$$Q(g) = \varprojlim (N^A_B g ; g^{-1}(B) \subset A \in \mathrm{Fin}\, A(X) ; B \in \mathrm{Fin}\, A(Y)) .$$

(8.2) Definition.

$$H(g) \overset{\mathrm{df}}{=} \varprojlim (H(N^A_B g) ; g^{-1}(B) \subset A \in \mathrm{Fin}\, A(X) ; B \in \mathrm{Fin}\, A(Y))$$

(in cohomological case there should be direct limit). Note that H is a functor ([1], 3.4).

§ 9. BOUNDARY OPERATOR, EXACTNESS, DIMENSION AXIOM

Let us now consider the boundary operator and exactness. Assume that (X, X_0) is a compact pair. Let $A \in \operatorname{Fin} A(X)$. Let $i: X_0 \to X$ and $j: X \to (X, X_0)$ be the canonical inclusion. Let $A \in \operatorname{Fin} A(X)$.

From the exactness of the combinatorial cubical homology we have an exact sequence:

$$\cdots \to H_k(N_{X_0}A; G) \to H_k(NA; G) \to H(NA, N_{X_0}A; G) \to H_{k-1}(NA; G) \to$$

Assume that $A_1 \in \operatorname{Fin} A(X)$ and that $A \subset A_1$. Then the following diagram commutes

$$\begin{array}{ccccccccc}
\cdots \to H_k(N_{X_0}A_1; G) & \to & H_k(NA_1; G) & \to & H_k(NA_1, N_{X_0}A_1; G) & \xrightarrow{\delta} & H_{k-1}(N_X A_1) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\cdots \to H_k(N_{X_0}A; G) & \to & H_k(NA; G) & \to & H_k(NA; N_{X_0}A; G) & \xrightarrow{\delta} & H_{k-1}(N_{X_0}A) \to
\end{array}$$

Passing to the limit we obtain the sequence:

$$(9.1) \quad \cdots \to H_k(X_0; G) \xrightarrow{H_k(i)} H_k(X; G) \xrightarrow{H_k(j)} H_k(X, X_0; G) \xrightarrow{\delta} H_{k-1}(X_0; G) \to$$

which is in general half-exact, in the sense that the composition of any two maps is zero. But if G is a compact topological group, or a vector space over a field then (9.1) is exact. [Look Eilenberg-Steenrod, " Foundations of algebraic topology ", VIII, 5.6].

The cohomological analogue of (9.1) is always exact.

Let $f: (X, X_0) \to (Y, Y_0)$ be a continuous map. Let $A \in \operatorname{Fin} A(X)$ $B \in \operatorname{Fin} A(Y)$ and $f^{-1}(B) \subset A$. Axiom 3 of the combinatorial theory shows that the diagram

$$\begin{array}{ccc}
H(NA(X, X_0)) & \xrightarrow{H(N_B^A f)} & H(NB(Y, Y_0)) \\
\downarrow & & \downarrow \\
H_{k-1}(N_{X_0}A) & \xrightarrow{H(N_B^A f)} & H_{k-1}(N_{Y_0}B)
\end{array}$$

commutes.

Passing to the limit we have the commutative diagram:

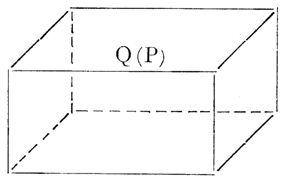
$$(9.2) \qquad \begin{array}{ccc}
H_k(X, X_0) & \xrightarrow{H_k(f)} & H_k(Y, Y_0) \\
\delta_k \downarrow & & \downarrow \delta_k \\
H_{k-1}(X_0) & \xrightarrow{H_{k-1}(f)} & H_{k-1}(Y_0)
\end{array}$$

(9.2) is the third Eilenberg-Steenrod axiom.

We will now discuss the dimension axiom. Let P be a point. Then $A(P)$ consists of three elements:

$$A_1 = (P, P) \quad ; \quad A_2 = (P, \varnothing) \quad \text{and} \quad A_3 = (\varnothing, P).$$

$Q(P) = NA(P) \subset I^3$



$NA(P)$ has one one-dimensional face given by:

$$\beta : \{A_2, A_3\} \to \{-1, 1\}$$

$$\beta(A_3) = -1$$

$$\beta(A_3) = 1$$

and two vertices: $(-1, -1, 1)$ and $(1, -1, 1)$. By (4.8)

(9.3) $$H_k(P) = 0 \quad \text{for} \quad k \neq 0.$$

(9.3) is the Eilenberg-Steenrod dimension axiom.

§ 10. Homotopy axiom. Excision axiom

(10.1) Lemma. *If $f, g : (X, X_0) \to (Y, Y_0)$ are continuous maps such that $Q(f)$ and $Q(g)$ are contiguous then* $H(f) = H(g)$.

Proof. It is a consequence of (II., 1.2′).

(10.2) Theorem (*Homotopy axiom*). *Let f and g be a pair of homotopic maps from* $(X, X_0) \to (Y, Y_0)$. *Then* $H(f) = H(g) : H(X, X_0) \to H(Y, Y_0)$.

Proof. By the Theorem 3.9 in [2] $Q(f)$ and $Q(g)$ are contiguous.

Let (X, X_0) be a compact pair. Let $U \subset X$ be an open subset of X such that $\overline{U} \subset \operatorname{int} X_0$. We have the canonical inclusion map $i : (X \setminus U, X_0 \setminus U) \to (X, X_0)$. Notice that there exists $G = (G_{-1}, G_1) \in A(X)$ such that $G_{-1} \subseteq \operatorname{Int} X_0$ and $G_1 \subseteq X \setminus \overline{U}$. Note that finite subsets $A(X)$ containing G are cofinal in $\operatorname{Fin} A(X)$. Furthermore, the family of all subsets of $A(X \setminus U)$ of the form $i^{-1}(A)$, where $G \in A \in \operatorname{Fin} A(X)$ are cofinal in $\operatorname{Fin} A(X \setminus U)$. Thus to show that

$$H(i) : H(X \setminus U, X_0 \setminus U) \to H(X, X_0)$$

is an isomorphism we need only to show that:

$$C_g(N_A^{i^{-1}(A)}\, i) : N_i^{-1}(A)(X \setminus U, X_0 \setminus U) \to NA(X, X_0) \tag{10.3}$$

is an isomorphism for finite subsets A of A (X) containing G. Let $\alpha : A \to i^{-1}(A)$ be the canonical map. Let us recall ([2], Theorem 2.6) that the following diagram commutes

$$\begin{tikzcd} N i^{-1}(A)(X \setminus U, X_0 \setminus U) \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & N_\alpha(X \setminus U, X_0 \setminus U) \arrow[d, "Id_A"] \\ & NA(X, X_0) \end{tikzcd} \tag{10.4}$$

(10.5) LEMMA. *If* G *is an element of* A *then*

$$C(Id_A) : C(N_\alpha(X \setminus U, X_0 \setminus U)) \to C(NA(X, X_0))$$

is an isomorphism in the category Ch.

Proof. Let $h \in C(Id_A)$.

i) Clearly h is an injection.

ii) Let F_β be a face of $NA(X, X_0)$ such that $F_\beta \neq 0$ $C(NA(X, X_0))$. Let $B \subset A$ be the domain of B. Note that $G \in B$ and $\beta(G) = 1$. [Otherwise, F_β would be zero in $C(NA(X, X_0))$, because $G_{-1} \cap G_1 \subset G_{-1} \subset X_0$]. For similar reasons

$$\cap \{G_{(a,\varepsilon)} : \varepsilon = \beta(a) \quad \text{or} \quad a \in A \setminus N\} \cap X_0 \neq \varnothing .$$

Hence F_β is also a non-zero chain in $C(N_\alpha(X \setminus U, X_0 \setminus U))$.

Passing to the contiguous level we obtain the commutative diagram:

$$\begin{tikzcd} C_g(N_i^{-1}(A)(X \setminus U, X_0 \setminus U)) \arrow[r, "C_g(\alpha)"] \arrow[dr, "C_g(N_A^{i^{-1}(A)}\, i)"'] & C_g(N_\alpha(X \setminus U, X_0 \setminus U)) \arrow[d, "C_g(Id_A)"] \\ & C_g(NA(X, X_0)) \end{tikzcd}$$

By ([2], Theorem 2.4) $Cg(\alpha)$ is an isomorphism. By (10.5) $Cg(Id_A)$ is an isomorphism. Hence $Cg(N_A^{i^{-1}(A)}\, i)$ is an isomorphism. This concludes the proof of (10.4). Thus we have:

(10.6) THEOREM (*Excision axiom*). *Let* (X, X_0) *be a compact pair and let* U *be an open subset of* X *such that* $\overline{U} \subset \operatorname{int} X_0$. *Let* $i : (X \setminus U, X_0 \setminus U) \to (X, X_0)$ *be the canonical inclusion. Then* $H(i) : H(X \setminus U, X_0 \setminus U) \to H(X, X_0)$ *is an isomorphism.*

§ 11. Continuity of cubical homology

Throughout this section $(X, p_t; t \in T)$ is a representation of a compact space X as limit of an inverse system $(X_t, p_t^s: t < S \in T)$ of compact spaces X_t, $t \in T$.

The proofs of the following lemmas are standard in the point set topology.

(11.1) Lemma. *For every finite* $A \subseteq A(X)$ *there exists* $t \in T$ *and a function* $f: A \to A(X_t)$ *such that* $p_t^{-1}(f(G)) \subseteq G$ *for every* $G \in A$.

(11.2) Lemma. *For every finite* $A \subseteq A(X_t)$ *there exists* $s > t$ *and a function* $f: A \to A(X_s)$ *such that*

i) $f(G) \subseteq (p_t^s)^{-1}(G)$

ii) $\cap \{\pi_\varepsilon(G) : (G, \varepsilon) \in Z\} = \varnothing$ *iff* $\cap \{\pi_\varepsilon(p_t^{-1}(G)) : (G, \varepsilon) \in Z\} = \varnothing$ *for every* $Z \subseteq f(A) \times \{-1, 1\}$.

(11.3) *Remark.* (*ii*) shows that p_s induces cubical isomorphism

$$N^{p_s^{-1}(f(A))}{}_{f(A)}(p_s) : Np_s^{-1}(f(A)) \to Nf(A) .$$

Assertions (11.1), (11.2), (11.3) and ([2], Theorem 3.4) imply:

(11.4) Corollary. *For every finite* $A' \subseteq A(X)$ *there exist* $t \in T$, $A \subseteq A(X_t)$ *and a function* $f: A' \xrightarrow{\text{onto}} A$ *such that*:

i) $p_t^{-1}(f(G)) \subseteq G$ *for every* $G \in A$.

ii) Morphism

$$N^{p_t^{-1}(A)}{}_{A}(p_t) : Np_t^{-1}(A) \to NA \quad \textit{is a contiguouos equivalence.}$$

iii) For $A_1 = A' \cup p_t^{-1}(A)$ *a morphism* $i^{p_t^{-1}(A)}{}_{A_1} : NA_1 \to Np_t^{-1}(A)$ *is a contiguous equivalence.*

Thus we have the following diagram.

$$\begin{tikzcd}
H(NA') & & & \\
& H(Np_t^{-1}(A)) \arrow[rr, "1-1"] & & H(NA) \\
H(NA_1) \arrow[uu] \arrow[ur, "1-1"] & & & \\
H(X) \arrow[u] \arrow[rrr, "H(p_t)"'] & & & H(X_t) \arrow[uu]
\end{tikzcd}$$

$H(NA')$, $H(NA)$, $H(NA_1)$ and $H(Np_t^{-1}(A))$ denote combinatorial homology of respective polyhedra.

As a corollary we have:

(11.5) THEOREM (*continuity*). *Cubical homology (and cohomology) theory is continuous.*

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