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A generalization of closed mappings

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Topologia. — *A generalization of closed mappings.* Nota di
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RIASSUNTO. — Se X ed Y sono spazi topologici, un sottoinsieme di X è detto *semichiuso* se esso è intermedio fra un insieme chiuso di X ed il suo interno; inoltre, un'applicazione $f: X \rightarrow Y$ è detta *semichiusa* se essa trasforma ogni insieme chiuso di X in un insieme semichiuso di Y. La presente Nota dà alcune caratterizzazioni di tali applicazioni.

1. INTRODUCTION

N. Levine [4] defined a subset A of a topological space X to be *semi-open* if there exists an open set U in X such that $U \subset A \subset \mathrm{Cl}U$, where ClU denotes the closure of U in X. He also defined a mapping f of a topological space X into a topological space Y to be *semi-continuous* if for any open set V in Y, $f^{-1}(V)$ is a semi-open set in X. Recently N. Biswas [3] defined a mapping $f: X \rightarrow Y$ to be *semi-open* if for any open set U in X, $f(U)$ is a semi-open set in Y. The purpose of the present note is to introduce a new class of mappings called *semi-closed* mappings which contains the class of closed mappings and to give some characterizations of such mappings.

Throughout the present Note X, Y and Z will always denote topological spaces on which no separation axioms are assumed. No mapping is assumed to be continuous otherwise stated. When A is a subset of a topological space X, the closure of A in X is denoted by ClA; the interior of A in X is denoted by IntA. Furthermore, we shall denote the family of all semi-open sets in X by SO(X).

2. SEMI-CLOSED MAPPINGS

DEFINITION 1. A subset A of a topological space X is said to be *semi-closed* if there exists a closed set F such that $\mathrm{Int}\, F \subset A \subset F$ [2].

DEFINITION 2. A mapping $f: X \rightarrow Y$ is said to be *semi-closed* if the image $f(F)$ of each closed set F in X is semi-closed in Y.

Remark 1. Every closed mapping is semi-closed, but the converse is false, as shown by the following example.

Example. Let $X = \{a, b, c\}$, $J_1 = \{\phi, \{a\}, \{a, b\}, \{a, c\}, X\}$ and $J_2 = \{\phi, \{a\}, \{a, b\}, X\}$. Let $f: (X, J_1) \rightarrow (X, J_2)$ be the identity mapping. Then it follows from $SO(X, J_2) = J_1$ that f is semi-closed, but f is not closed.

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LEMMA (Biswas [2]). *The following three properties of a subset* A *of* X *are equivalent*:

(1) A *is semi-closed.*
(2) $\mathrm{Int}\,\mathrm{Cl}A \subset A$.
(3) $X - A$ *is semi-open.*

D. R. Anderson and J. A. Jensen [1] showed that if $f: X \to Y$ is a continuous and open mapping, then $f^{-1}(B) \in SO(X)$ for every $B \in SO(Y)$. The following theorem is a slight improvement of this theorem.

THEOREM 1. *If* $f: X \to Y$ *is an open and semi-continuous mapping, then* $f^{-1}(B) \in SO(X)$ *for every* $B \in SO(Y)$.

Proof. Suppose B is an arbitrary semi-open set in Y. Then there exists an open set V in Y such that $V \subset B \subset \mathrm{Cl}V$. By the openness of f, we have $f^{-1}(V) \subset f^{-1}(B) \subset f^{-1}(\mathrm{Cl}V) \subset \mathrm{Cl}[f^{-1}(V)]$ [5, (i), p. 13]. Since f is semi-continuous and V is open in Y, $f^{-1}(V) \in SO(X)$. Therefore by Theorem 3 of [4], we obtain $f^{-1}(B) \in SO(X)$.

COROLLARY. *If* $f: X \to Y$ *is an open and semi-continuous mapping, then the inverse image* $f^{-1}(B)$ *of each semi-closed set* B *in* Y *is semi-closed in* X.

Proof. This follows immediately from Theorem 1 and Lemma.

Remark 2. The composition mapping of two semi-closed mappings is not always semi-closed, as shown by Example 7 of [3] and Lemma.

THEOREM 2. *Let* $f: X \to Y$ *and* $g: Y \to Z$ *be two mappings, and let* $g \circ f: X \to Z$ *is a semi-closed mapping. Then*:

(1) *If* f *is continuous and surjective, then* g *is semi-closed.*
(2) *If* g *is open, semi-continuous and injective, then* f *is semi-closed.*

Proof. In order to prove the statement (1) suppose H is an arbitrary closed set in Y. Then $f^{-1}(H)$ is closed in X because f is continuous. Since $g \circ f$ is semi-closed and f is surjective, $(g \circ f)[f^{-1}(H)] = g[f\{f^{-1}(H)\}] = g(H)$ is semi-closed in Z. This implies that g is a semi-closed mapping. In order to prove the statement (2) suppose F is an arbitrary closed set in X. Then $(g \circ f)(F)$ is semi-closed in Z because $g \circ f$ is semi-closed. Since g is injective, we have $g^{-1}[(g \circ f)(F)] = f(F)$. It follows immediately from Corollary that $f(F)$ is a semi-closed set in Y because g is open and semi-continuous. This implies that f is semi-closed.

3. CHARACTERIZATIONS

THEOREM 3. $f: X \to Y$ *is a semi-closed mapping if and only if* $f(\mathrm{Cl}A) \supset \mathrm{Int}\,\mathrm{Cl}[f(A)]$ *for every subset* A *of* X.

Proof. Necessity. Suppose f is a semi-closed mapping and A is an arbitrary subset of X. Then $f(\mathrm{Cl}A)$ is semi-closed in Y. By Lemma, we obtain $f(\mathrm{Cl}A) \supset \mathrm{Int}\,\mathrm{Cl}\,[f(\mathrm{Cl}A)] \supset \mathrm{Int}\,\mathrm{Cl}\,[f(A)]$. The proof of this part is complete.

Sufficiency. Suppose F is an arbitrary closed set in X. Then by hypothesis, we have $\mathrm{Int}\,\mathrm{Cl}\,[f(F)] \subset f(\mathrm{Cl}F) = f(F)$. It follows from Lemma that $f(F)$ is semi-closed in Y. This implies that f is semi-closed.

DEFINITION 3. The intersection of all semi-closed sets containing a subset A of X is said to be the *semi-closure* of A and is denoted by $\mathrm{Cs}(A)$ [2].

THEOREM 4. *$f: X \rightarrow Y$ is a semi-closed mapping if and only if $\mathrm{Cs}\,[f(A)] \subset f(\mathrm{Cl}A)$ for every subset A of X.*

Proof. Necessity. Suppose f is semi-closed and A is an arbitrary subset of X. Then $f(\mathrm{Cl}A)$ is semi-closed in Y. Since $f(A) \subset f(\mathrm{Cl}A)$, we obtain $\mathrm{Cs}\,[f(A)] \subset f(\mathrm{Cl}A)$.

Sufficiency. Suppose F is an arbitrary closed set in X. By the hypothesis, we obtain $f(F) \subset \mathrm{Cs}\,[f(F)] \subset f(\mathrm{Cl}F) = f(F)$ and hence $f(F) = \mathrm{Cs}\,[f(F)]$. By Theorem 3 of [2], $f(F)$ is semi-closed in Y. This implies that f is semi-closed.

It is well known that if $f: X \rightarrow Y$ is a closed mapping, then for each point y in Y and each open set U in X containing $f^{-1}(y)$ there exists an open set V in Y containing y such that $f^{-1}(V) \subset U$. The following theorem is a generalization of this theorem.

THEOREM 5. *A surjective mapping $f: X \rightarrow Y$ is semi-closed if and only if for each subset B in Y and each open set U in X containing $f^{-1}(B)$, there exists a semi-open set V in Y containing B such that $f^{-1}(V) \subset U$.*

Proof. Necessity. Suppose B is an arbitrary subset in Y and U is an arbitrary open set in X containing $f^{-1}(B)$. We put $V = Y - f(X - U)$. Then by Lemma, V is semi-open in Y. Since $U \supset f^{-1}(B)$, it follows from a straightforward calculation that $V \supset B$. Moreover, we have $f^{-1}(V) = X - f^{-1}[f(X - U)] \subset U$. The proof of this part is complete.

Sufficiency. Suppose F is an arbitrary closed set in X. Let y be an arbitrary point in $Y - f(F)$, then $f^{-1}(y) \subset X - f^{-1}[f(F)] \subset X - F$ and $X - F$ is open in X. Hence by the hypothesis, there exists a semi-open set V_y containing y such that $f^{-1}(V_y) \subset X - F$. This implies that $y \in V_y \subset Y - f(F)$. By Theorem 2 of [4], we obtain that $Y - f(F) = \cup \{V_y \mid y \in Y - f(F)\}$ is semi-open in Y. Therefore, by Lemma, $f(F)$ is semi-closed. This shows that f is semi-closed.

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