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On the Indicatrix and Isotropy Group in Finsler Spaces with Lorentz Signature

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Geometria. — *On the Indicatrix and Isotropy Group in Finsler Spaces with Lorentz Signature.* Nota di JOHN K. BEEM, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si studia l'indicatrice di uno spazio di Finsler indefinito con segnatura di Lorentz e si mostra che, per spazi di dimensione ≥ 3, la parte positiva dell'indicatrice possiede una sola componente. Si prova poi che i moti di un tale spazio costituiscono un gruppo di Lie e si assegnano condizioni sufficienti affinchè il gruppo di isotropia ammetta una rappresentazione fedele quale sottogruppo del gruppo di Lorentz.

Per eventuali applicazioni alla fisica dei risultati qui ottenuti cfr. [4].

1. INTRODUCTION

The n–dimensional Finsler Manifolds which have a metric tensor with Lorentz signature have been investigated in [1]. In [2] the motions of two dimensional indefinite Finsler Spaces have been studied. The present article is concerned with the indicatrix and isotropy group in higher dimensions. This article may be of some interest to physicists in connection with the study of space-times using Finsler spaces rather than Riemannian spaces. Kordo [4] has investigated a Finslerian approach to space-times.

Let M be an n–dimensional indefinite Finsler manifold and let $T'(M)$ denote the reduced tangent bundle of M which consists of the tangent bundle $T(M)$ less the zero vectors. If M has signature $n - 2s$ then $T'(M)$ has signature $2n - 4s$. Consideration of the metric on $T'(M)$ shows that the group of motions of M is a Lie Group.

When M has the Lorentz signature and dimension n at least three we show that the positive part of the indicatrix at each point has exactly one component. This implies the whole indicatrix must have an odd number of components for $n \geq 3$. Examples are given which show that there are spaces with Lorentz signature such that for $n = 2$ the indicatrix has more than four components and for $n \geq 3$ there are spaces in which the indicatrix has more than the expected three components.

Wang [10] has shown that for definite Finsler spaces the isotropy subgroup $I_x(M)$ of the motions of M always has a faithful representation as a subgroup of the orthogonal group. One would therefore expect that for Finsler manifolds of Lorentz signature the isotropy group would always have a faithful representation as a subgroup of the Lorentz group. We give a counter example which shows this is not true in general.

(*) Nella seduta del 10 marzo 1973.

However, we are able to show that when M has the Lorentz signature, the light cone is elliptic and the indicatrix has exactly three components then the isotropy group $I_x(M)$ always admits a representation as a subgroup of the Lorentz group.

2. Indefinite Finsler Manifolds

Let M be an n–dimensional connected and paracompact differentiable manifold of class C^∞. Denote the local coordinates of a point x by $x^1, x^2, \cdots, x^n$ and let $T(x)$ denote the tangent space at x. In $T(x)$ we use the natural frame $\partial/\partial x^1, \cdots, \partial/\partial x^n$ and for a vector y in $T(x)$ let $y^1, \cdots, y^n$ be the components of y in the natural basis. Let $L(x, y)$ be a continuous function defined on the tangent bundle T(M) of M which has the following properties:

(A) The function $L(x, y)$ is of class C^∞ if $y \neq 0$.

(B) $L(x, ky) = k^2 L(x, y)$ for all $k > 0$.

(C) The metric tensor $g_{ij}(x, y) = \frac{1}{2} \partial^2 L/\partial y^i \partial y^j$ has s negative and $n - s$ positive eigenvalues for all (x, y) with $y \neq 0$.

(D) $|L(x, -y)| = |L(x, y)|$.

The function $L(x, y)$ will be called the basic metric function. It corresponds to the square of the fundamental function which is usually studied in definite Finsler spaces. The manifold M with the function $L(x, y)$ is called an indefinite Finsler space of signature $n - 2s$. If the basic metric function $L(x, y)$ is replaced with $-L(x, y)$, then M becomes a space of signature $2s - n$.

When $s = 0$ the space M is a definite Finsler space, compare [7]. If $s = 1$, (or $s = n - 1$), then M has the Lorentz signature. In this paper we consider $1 \leq s \leq n - 1$ in Sections 1, 2 and 3. In the rest of the paper we only consider $s = 1$.

The pseudo–Riemannian manifolds are those in which the metric tensor $g_{ij}(x, y)$ depends only on position x. When $g_{ij}(x, y)$ depends only on direction y we say that M is locally Minkowskian. If M is R^n and is locally Minkowskian we call M simply a Minkowskian space. A Minkowskian space which is also pseudo–Riemannian is called pseudo–Euclidean. The above definition of Minkowskian space differs from the one generally used by physicists. In physics Minkowskian space refers to the four dimensional pseudo–Euclidean space of signature two.

3. Motions

For each fixed point x_0 of M there is an induced Minkowskian metric on $T(x_0)$ with basic metric function $L(x_0, y)$. Let M and $\bar{M}$ be indefinite Finsler manifolds with basic metric functions $L(x, y)$ and $\bar{L}(x', y')$ respectively. A diffeomorphism $f: M \to \bar{M}$ is called an isometry of M onto $\bar{M}$ if each of the tangent maps $(f_*)_x : \mathrm{T}(x) \to \overline{\mathrm{T}}(f(x))$ preserves the indefinite metric on the tangent spaces.

When $\mathrm{M} = \overline{\mathrm{M}}$ we say f is a motion of M. The group of motions of M is denoted I(M). The subgroup of I(M) which leaves the point x fixed is called the isotropy group at x and is denoted $\mathrm{I}_x(\mathrm{M})$.

Sasaki [8] has shown how to define an associated Riemannian metric on the tangent bundle of a Riemannian manifold. The Sasaki method may be used to define a Riemannian metric on $\mathrm{T}'(\mathrm{M})$ given a definite Finsler metric on M. This has been studied in [5], [6] and [11]. The same method applied equally well to indefinite Finsler spaces and we will not detail the method in this paper. It is clear (compare [6, p. 184]) that if M is a Finsler space of signature $n - 2s$ then $\mathrm{T}'(\mathrm{M})$ becomes a pseudo-Riemannian space of signature $2n - 4s$. Furthermore, any motion $f : \mathrm{M} \to \mathrm{M}$ has a naturally associated map $f_* \mid \mathrm{T}'(\mathrm{M}) : \mathrm{T}'(\mathrm{M}) \to \mathrm{T}'(\mathrm{M})$ which is a motion of $\mathrm{T}'(\mathrm{M})$. We will use f_* to denote $f_* \mid \mathrm{T}'(\mathrm{M})$ in the next lemma.

LEMMA 1. *The group* I(M) *of motions of* M *is a Lie Group.*

Proof. The space $\mathrm{T}'(\mathrm{M})$ is pseudo-Riemannian and consequently its group of motions $\mathrm{I}(\mathrm{T}'(\mathrm{M}))$ is a Lie Group. The correspondence $f \to f_*$ is an isomorphism of I(M) onto a closed subgroup of $\mathrm{I}(\mathrm{T}'(\mathrm{M}))$. Since a closed subgroup of a Lie Group is a Lie Group the lemma is established.

4. THE INDICATRIX

In the rest of this paper we assume M has signature $n - 2$. If x is a fixed point of the manifold M we define the indicatrix K_x and light cone C_x as the following subsets of the tangent space at x

$$\mathrm{K}_x^+ = \{ y \in \mathrm{T}(x) \mid \mathrm{L}(x, y) = 1 \}$$

$$\mathrm{K}_x^- = \{ y \in \mathrm{T}(x) \mid \mathrm{L}(x, y) = -1 \}$$

$$\mathrm{K}_x = \mathrm{K}_x^+ \cup \mathrm{K}_x^-$$

$$\mathrm{C}_x = \{ y \in \mathrm{T}(x) \mid \mathrm{L}(x, y) = 0 \} .$$

Properties (B) and (D) imply the set C_x consists of a union of lines through the origin of $\mathrm{T}(x)$. The sets K_x^+ and K_x^- are called respectively the positive indicatrix and negative indicatrix. If M is a pseudo-Riemannian manifold of Lorentz signature, then at each point x the set K_x has exactly four components if $n = 2$ and exactly three components if $n \geq 3$. On the other hand if M is a Finsler space of Lorentz signature the set K_x may have more than four components. Below we give two examples. In the first example K_x has eight components at each point and in the second K_x has five components. The first example is a two dimensional Minkowskian space and the second is an n-dimensional Minkowskian space for $n \geq 3$.

Example 1. Let $M = R^2$ and define

$$L(x, y) = \frac{(y^1)^3 y^2 - y^1 (y^2)^3}{(y^1)^2 + (y^2)^2}.$$

Example 2. Let $M = R^n$ and define (for sufficiently large α and $n \geq 3$)

$$L(x, y) = \frac{2 (y^1)^3 y^2}{\Sigma (y^i)^2} - y^1 \cdot y^2 + .4 (y^1)^2 - .4 (y^2)^2 + \alpha \left(\sum_{j=3}^{n} (y^j)^2 \right).$$

It has been shown [1, p. 1036] that the negative components of K_x are strictly convex hypersurfaces in $T(x)$. In the next lemma we consider $C_x - \{x\}$ where x is used to denote both a fixed point of M and the zero vector in $T(x)$. We recall that M has signature $n-2$.

LEMMA 2. *Let $n \geq 3$. The set $C_x - \{x\}$ has exactly as many components as K_x^-. The closure of each component of $C_x - \{x\}$ is a convex hypersurface in $T(x)$ and is differentiable at each point except x. If y_0 is a point of some component* S, *then the tangent hyperplane to* S *at y_0 intersects* S *in a half line.*

Proof. The fact that $C_x - \{x\}$ has exactly as many components as K_x^- follows from lemma 1 and lemma 8 of [1].

Let S be a component of $C_x - \{x\}$. The conditions (A) and (C) imposed on $L(x, y)$ imply that S is a differentiable hypersurface in $T(x)$. Lemma 1 of [1] implies that the closure of S is the boundary of a convex set.

Let $y_0 \in S$. The formula for the normal curvature k_n of S in a direction y in the tangent hyperplane to S at y_0 is given in [1, p. 1036]. Consideration of this formula yields that at y_0 the surface S has $n-2$ positive principal curvatures and one principal curvature of value zero. It now follows that the tangent hyperplane at y_0 intersects S only in a half line.

THEOREM 3. *If $n \geq 3$, then K_x^+ has exactly one component.*

Proof. Let $(y^1, \cdots, y^n)$ be coordinates in $T(x)$ and define $S_x = \{(y^1, \cdots, y^n) \mid \Sigma (y^i)^2 = 1\}$. Set $N_x = C_x \cap S_x$. Lemma 2 implies that N_x has a finite number of components and each component is topologically an $(n-2)$ sphere which is the boundary in S_x of a topological $(n-1)$ ball. Furthermore, each component of N_x is strictly convex with respect to the great circles of S_x.

Consider the map of K_x^+ into S_x defined by $y \to \lambda y$ where $\lambda = (\Sigma (y^i)^2)^{-1/2}$. This is a diffeomorphism of K_x^+ onto an open subset of the manifold S_x. The image of the map is S_x less the closure of the $(n-1)$ balls which the components of N_x bound. It follows that K_x^+ must be connected. This establishes the theorem.

COROLLARY 4. *If $n \geq 3$, then the indicatrix* K *has an odd number of components.*

Proof. We need only show that K_x^- has an even number of components. This follows from condition (D) and Lemma 1 of [1] when $n \geq 3$.

It should be noted that for $n \geq 3$ (and Lorentz signature) condition (D) reduces to $L(x, -y) = L(x, y)$. An example given in [1, p. 1039] shows that condition (D) does not reduce to the above form for $n - 2$. Example 1 of the present paper shows that Theorem 3 and corollary 4 are not valid for $n = 2$.

LEMMA 5. *Let x_1 and x_2 be two points of the (connected) manifold* M. *The indicatrix at x_1 and the indicatrix at x_2 have the same number of components.*

Proof. Let $x(t)$ for $0 \leq t \leq 1$ be a curve on M with $x(0) = x_1$ and $x(1) = x_2$. Let $n(t)$ be the number of components of $K_{x(t)}$. It is sufficient to show that for each t_0 there is a relatively open interval about t_0 in which $n(t)$ is constant. This follows from the continuity of $L(x, y)$ and the fact that each component of $C_x - \{x\}$ bounds a convex set with interior points.

5. CONVEX SURFACES

A (strictly) convex hypersurface S will be a connected set which is the boundary in R^n of its convex hull which is required to be a closed (strictly) convex set with interior points. Let O denote the origin of R^n. If O is not in the convex hull of S, then the light cone C(S) of S is defined to be the boundary of the set $\{y \in R^n \mid y = \lambda \bar{y} \text{ for } \bar{y} \in S \text{ and } \lambda \in R^1\}$. The light cone of the convex hypersurface S consists of a union of lines each containing O. The light cone C(S) is elliptic if there is a hyperplane H of R^n which intersects each generator of C(S) and is such that $H \cap C(S)$ an $(n-2)$–dimensional ellipsoid.

Consider now the quadratic form $Q(y) = -(y^1)^2 + (y^2)^2 + \cdots + (y^n)^2$ defined for all $y \in R^n$. The Lorentz group $O^1(n)$ is the set of linear transformations which leave $Q(y)$ invariant. This group is the isotropy group of the n–dimensional Lorentz space. In the rest of this paper $Q(y)$ will always denote the above form.

LEMMA 6. *Let* S *be a convex hypersurface in* R^n *and assume that* O *does not belong to the convex hull of* S. *If the light cone* C(S) *is elliptic, then the group of linear transformations which map* S *onto itself has a faithful representation as a subgroup of the Lorentz group* $O^1(n)$.

Proof. By, if necessary, changing coordinates we may assume that the light cone C(S) is given by $Q(y) = 0$. Since S is mapped onto itself the light cone C(S) must always be mapped onto itself. The hyperboloid of two sheets $Q(y) = -a^2$ must be mapped onto the hyperboloid $Q(y) = -b^2$.

Let T be a linear transformation taking S onto S. Then T has a matrix representation A relative to the standard basis of R^n. The transformation must be non-singular since the convex hull of S is required to have interior points. We represent T by the matrix $\rho(T)$ defined by

$$\rho(T) = |\det A|^{-1/n} A.$$

Since det $\rho(\mathrm{T}) = \pm 1$, it follows that the transformation $\rho(\mathrm{T})$ represents maps $\mathrm{Q}(y) = -a^2$ onto $\mathrm{Q}(y) = -a^2$ and in fact preserves $\mathrm{Q}(y)$. It is clear that the representation is faithful.

In the above proof it is not hard to show that if S is compact the matrix A has determinant ± 1. The following example stated by Busemann in [3, p. 40] shows that when S is not compact the above matrix A may have det $\mathrm{A} \neq \pm 1$.

Example 3. Let S be the subset of R^2 given by $(y^1)^{1-\mu}(y^2)^{\mu} = 1$ and $y^1 > 0, y^2 > 0$ with $0 < \mu < 1$. If $\mu \neq 1/2$, then $\det \mathrm{A} \neq \pm 1$.

6. The Isotropy Group

It is easy to check that the isotropy group of the indefinite Finsler space in Example 1 has eight elements. Since the two dimensional Lorentz group has no subgroup of order eight, the isotropy group of Example 1 cannot have a faithful representation as a subgroup of $\mathrm{O}^1(2)$.

Let f be represented in local coordinates by $f(x) = (f^1(x), f^2(x), \cdots, f^n(x))$. Then the Jacobian matrix $\mathrm{J}(f)$ of f is the matrix

$$\left[\frac{\partial f^i}{\partial x^j}\right].$$

It is not hard to show that if M is a (connected) indefinite Finsler manifold and $x \in \mathrm{M}$, then the association of $f \in \mathrm{I}_x(\mathrm{M})$ with the Jacobian matrix $\mathrm{J}(f)$ is a faithful representation of $\mathrm{I}_x(\mathrm{M})$ as a subgroup of the general linear group $\mathrm{GL}(n)$.

In the next theorem we let K_1 and K_2 be components of K_x^-. We say K_1 and K_2 are opposed if $y \in \mathrm{K}_1$ implies $-y \in \mathrm{K}_2$. The light cone $\mathrm{C}(\mathrm{K}_1)$ is a subset of the light cone C_x in $\mathrm{T}(x)$.

THEOREM 7. *Let* M *have the Lorentz signature. Let* K_1 *and* K_2 *be opposed components of the negative indicatrix* K_x^-. *If the light cone* $\mathrm{C}(\mathrm{K}_1)$ *is elliptic then the group of* $f \in \mathrm{I}_x(\mathrm{M})$ *such that* $f_*(\mathrm{K}_1 \cup \mathrm{K}_2) = \mathrm{K}_1 \cup \mathrm{K}_2$ *has a faithful (matrix) representation as a subgroup of the Lorentz group* $\mathrm{O}^1(n)$. *The representation is given in correct local coordinates by* $\rho(f) = \mathrm{J}(f)$.

Proof. Let f_* denote the differential map of f restricted to $\mathrm{T}(x)$. Then Lemma 6 and the fact that K_1 and K_2 are opposed implies that there are local coordinates $(x^1, \cdots, x^n)$ such that

$$\rho(f) = |\det \mathrm{J}(f)|^{-1/n}\, \mathrm{J}(f)$$

is a faithful representation of the group as a subgroup of $\mathrm{O}^1(n)$.

We wish to show that det $\mathrm{J}(f) = \pm 1$. Consider $\mathrm{S}_x = \{y \in \mathrm{T}(x) \mid \Sigma (y^i)^2 = 1\}$ and define $\mathrm{D} = \mathrm{S}_x \cap \mathrm{C}(\mathrm{K}_1)$. The set D is compact. For $p \in \mathrm{D}$ expand in a power series using $h = (h^1, \cdots, h^n)$, $|h| = (\Sigma (h^i)^2)^{1/2}$ and $\mathrm{L}(x, p) = 0$.

$$\mathrm{L}(x, p + h) = \Sigma \mathrm{L}_{y^i}(x, p)\, h^i + \mathrm{O}(|h|^2).$$

Restrict h to be in the direction of the inward normal u_p to $C(K_1)$. Then $h = |h| u_p$ where $u_p^i = -L_{y^i}(x, p) [\Sigma L_{y^i}^2(x, p)]^{-1/2}$. Lemma 8 of [1] and the compactness of D imply there are positive numbers δ, m_1, and m_2 such that

$$-m_1 |h| \leq L(p + h) \leq -m_2 |h|$$

for all $h = |h| u_p$, all $p \in D$ and $\delta > |h| > 0$. This implies that there is some number $b \neq 0$ such that the hyperboloid of two sheets given by $Q(y) = -b^2$ lies in the interior of the union of the convex hulls of K_1 and K_2.

Assume now that $\det J(f) \neq \pm 1$ then by, if necessary, replacing f by f^{-1} we may assume $|\det J(f)| > 1$. The map f_* then takes a hyperboloid $Q(y) = -a_0^2$ onto another $Q(y) = -a_1^2$ where $a_1^2 > a_0^2$. Choose a_0 such that the hyperboloid $Q(y) = -a_0^2$ has at least one point in common with $K_1 \cup K_2$. Then there is an m such that f_* composed with itself m times takes $Q(y) = -a_0^2$ onto $Q(y) = a_m^2$ where $a_m^2 > b^2$. This contradicts the fact that $Q(y) = -a_m^2$ must have a point in common with $K_1 \cup K_2$.

It is possible that the above result remains valid without the assumption that $C(K_1)$ be elliptic. When $n = 2$ the set $C(K_1)$ consists of two lines and is trivially elliptic.

COROLLARY 8. *Let* M *be an* n*–dimensional manifold with the Lorentz signature. If* $n = 2$ *let the indicatrix* K_x *have exactly four components. If* $n \geq 3$ *let the indicatrix have exactly three components. Assume that the light cone* C_x *is elliptic. Then* $I_x(M)$ *has a faithful representation as a subgroup of* $O^1(n)$.

Proof. The indicatrix K_x has exactly two negative components K_1 and K_2. Hence $f_*(K_1 \cup K_2) = K \cup K_2$ and Theorem 6 implies the result.

COROLLARY 9. *Let* M *and* K_x *satisfy the hypothesis of Corollary 8. Furthermore, assume that* I (M) *is transitive on* M. *Then we can introduce a pseudo–Riemannian metric (with Lorentz signature) on* M *such that* I (M) *is a group of motions with respect to this new metric.*

Proof. From Corollary 8 it follows that a pseudo–Euclidean metric of signature $n - 2$ may be defined on $T(x)$ such that if $f \in I_x(M)$ then f_* restricted to $T(x)$ is a motion of this pseudo–Euclidean space. A pseudo–Riemannian metric on M may now be defined using the new metric on $T(x)$ and the transitivity of I (M).

Corollary 8 is the indefinite analogue to a result given in [9, p. 59]. It implies that the problem of determining the indefinite Finsler spaces which have the right number of components in each indicatrix, have elliptic light cones and which admit the transitive group of motions I (M) is reduced to determining the Lorentz manifolds which admit I (M) as a group of motions.

The final result gives a sufficient condition for the light cone to be elliptic.

THEOREM 10. *Let* $I_x(M)$ *be transitive on the generators of the light cone* $C(K_1)$. *Then* $C(K_1)$ *is elliptic.*

Proof. By Theorem 9 of [3, p. 43] it is only necessary to show that a cross section of $C(K_1)$ has an Euler point with non-vanishing Gauss Curvature. But this follows from the normal curvature k_n of $C(K_1)$ mentioned in the proof of Lemma 2.

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