Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

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On Cesàro-Nörlund summability of Jacobi Series

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.3, p. 354–364.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_3_354_0>

Analisi matematica. — *On Cesàro–Nörlund summability of Jacobi Series.* Nota di R. S. CHOUDHARY, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Si dimostra un teorema sulla sommabilità di *Cesàro–Nörlund* dello sviluppo di una funzione definita in [—1, 1] in serie di polinomi di Jacobi nei punti 1 e —1.

INTRODUCTION

In a previous paper published in this Journal (Choudhary, 1972), I have proved a theorem on the Nörlund summability of the Jacobi Series at the end point $x = 1$. The object of present paper is to prove the same theorem under less stringent conditions by superimposing the Nörlund means on Cesàro means of order one.

Let Σa_n be a given infinite series with the sequence of partial sums $\{S_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex and let us write

$$P_n = p_0 + p_1 + p_2 + \cdots + p_n .$$

The sequence to sequence transformation; viz.,

$$t_n = \frac{1}{P_n} \sum_{\nu=0}^{\nu=n} p_{n-\nu} S_\nu = \frac{1}{P_n} \sum_{\nu=0}^{\nu=n} p_\nu S_{n-\nu} , \quad P_n \neq 0 , \tag{1.1}$$

defines the sequence $\{t_n\}$ of Nörlund means of the sequence $\{S_n\}$, generated by the sequence of constants $\{p_n\}$. The series Σa_n is said to be summable (N, p_n) to sum S if $\lim t_n$ exists and equals S.

If the method of summability (N, p_n) is superimposed on the Cesàro means of orders one, a method of summability $(N, p_n) \cdot (C, 1)$ is obtained [6]. If σ_n denote the $(C, 1)$ mean of the sequence $\{S_n\}$, then (N, p_n). $(C, 1)$ mean of $\{S_n\}$ is given by

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} p_{n-k} \sigma_k . \tag{1.2}$$

The sequence $\{S_n\}$ is said to be summable $(N, p_n) \cdot (C, 1)$ to the sum S, if $\lim T_n$ exists and equals S.

The conditions of regularity of the method of summability (N, p_n) are

$$\lim_{n \to \infty} \frac{p_n}{P_n} = 0 \tag{1.3}$$

$$\lim_{n \to \infty} \sum_{k=0}^{k=n} |p_k| = O\,|P_n| . \tag{1.4}$$

(*) Nella seduta del 10 marzo 1973.

Two important particular cases of (N, p_n) summability are (*i*) Harmonic summability when $p_n = \frac{1}{n+1}$ and (*ii*) Cesàro summability when

$$p_n = \binom{n+\delta-1}{\delta-1}, \quad \delta > 0 .$$

2. The Fourier Jacobi expansion of a function $f(x) \in L[-1, 1]$ is given by

$$f(x) \sim \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) , \tag{2.1}$$

where

$$a_n = \frac{(2n+\alpha+\beta+1)\,\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)}{2^{\alpha+\beta+1}\,\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)} \times$$

$$\times \int_{-1}^{1} (1-x)^{\alpha}(1+x)^{\beta} f(x)\, P_n^{(\alpha,\beta)}(x)\, dx$$

and $P_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomials of order (α, β), $\alpha > -1$, $\beta > -1$. The Nth partial sum of the series (2.1) at the end point $x = +1$ is given by (Obrechkoff, [2], p. 99)

$$S_n(1) = 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} f(\cos\varphi)\, P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi .$$

Consequently,

$$S_n(1) - A = 2^{\alpha+\beta+1} \lambda_n \int_0^{\pi} \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} \times \tag{2.2}$$

$$\times [f(\cos\varphi) - A]\, P_n^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi .$$

Cesàro summability of the series (2.1) at the end point $x = +1$ has been studied by Kogbetliantz [1], Szegö [5], Obrechkoff [2] and Pandey [7]. Recently Gupta [3] and Author [4] have discussed the Nörlund summability of the series (2.1) at $x = +1$. Writing

$$F(\varphi) = [f(\cos\varphi) - A] \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} . \tag{2.3}$$

the Author has proved the following theorem:

THEOREM A. *Let* (N, p_n) *be a regular Nörlund method defined by a real non negative, monotonic non increasing sequence of coefficients* $\{p_n\}$, *such that* $P_n \to \infty$, *as* $n \to \infty$ *then if*

$$F_1(t) = \int_0^t |F(\varphi)|\, d\varphi = o\left[\frac{p(1/t)\, t^{2\alpha+1}}{P(1/t)}\right], \quad \text{as} \quad t \to 0 \tag{2.4}$$

and

$$\sum_{n} \frac{P_n}{n^{\alpha+1/2}} < \infty . \tag{2.5}$$

then the series (2.1) *is summable* (N, p_n) *at the point* $x = +1$ *to the sum* A *provided* $-1/2 \leq \alpha < 1/2$, $\beta > -1/2$, *and the antipole condition*

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4} \, |f(x)| \, dx < \infty , \tag{2.6}$$

b fixed, is satisfied.

The object of present paper is to prove above theorem under less stringent conditions by superimposing the Nörlund means on the Cesàro means of order one.

Writing

$$\psi(\varphi) = [f(\cos\varphi) - A] , \tag{2.7}$$

we prove the following theorem:

THEOREM. *If* (N, p_n) *is a Nörlund method defined by a real non negative monotonic non increasing sequence of coefficients* $\{p_n\}$, *then if*

$$\Psi(t) = \int_{0}^{t} |\psi(\varphi)| \, d\varphi = o(t) , \qquad t \to 0 , \tag{2.8}$$

then the series (2.1) *is summable* $(N, p) \cdot (C, 1)$ *at the point* $x = 1$ *to the sum* A, *provided* $-1/2 < \alpha < 1/2$, $\beta > -1/2$ *and the antipole condition*

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4} \, |f(x)| \, dx < \infty , \tag{2.9}$$

b fixed, is satisfied.

In view of the relation

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n \, P_n^{(\beta,\alpha)}(x)$$

the corresponding problem at the point $x = -1$, *can be solved in the similar manner.*

3. For the proof of the theorem we shall require the following lemmas.

LEMMA 1 (Szegö [5], p. 167). *For* α, β *arbitrary and real and c a fixed positive constant then*

$$P_n^{(\alpha,\beta)}(\cos\theta) = \begin{cases} \theta^{-\alpha-1/2} \, O(n^{-1/2}) , & c/n \leq \theta \leq \pi/2 \\ O(n^{\alpha}) , & 0 \leq \theta \leq c/n . \end{cases} \tag{3.1}$$

LEMMA 2 (Szegö [5], p. 190). *If* $\alpha > -1$, $\beta > -1$, $c/n \leq \theta \leq \pi - c/n$ *then*

$$P_n^{(\alpha,\beta)}(\cos\theta) = n^{-1/2}\, k(\theta)\left[\operatorname{Cos}(N\theta+\gamma) + \frac{O(1)}{n \operatorname{Sin}\theta}\right], \tag{3.2}$$

where

$$k(\theta) = \frac{1}{\sqrt{\pi}}\left(\operatorname{Sin}\frac{\theta}{2}\right)^{-\alpha-1/2}\left(\operatorname{Cos}\frac{\theta}{2}\right)^{-\beta-1/2} \tag{3.3}$$

and

$$N = n + \frac{\alpha+\beta+1}{2}, \quad \gamma = -\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}. \tag{3.4}$$

LEMMA 3 (Gupta [3]). *The antipole condition* (2.9), *viz., the condition*

$$\int_{-1}^{b} (1+x)^{\beta/2-3/4}\,|f(x)|\,dx < \infty,$$

implies for $\beta > -1/2$

$$\int_{a=\cos^{-1} b}^{\pi} |f(\cos\theta) - A|\left(\cos\frac{\theta}{2}\right)^{\beta-1/2} d\theta < \infty, \tag{3.5}$$

which further implies

$$\int_0^{1/n} |f(-\cos t) - A|\,(t^{\beta-1/2})\,dt = o(1). \tag{3.6}$$

LEMMA 4. *Let*

$$M_{n-k}(\varphi) = \frac{1}{(n+1-k)}\sum_{m=0}^{m=n-k} \lambda_m\, P_m^{(\alpha+1,\beta)}(\cos\varphi) \tag{3.7}$$

and

$$N_n(\varphi) = \frac{2^{\alpha+\beta+1}}{P_n}\sum_{k=0}^{n} p_k\, M_{n-k}(\varphi), \tag{3.8}$$

where

$$\lambda_n = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\,\frac{\Gamma(n+\alpha+\beta+2)}{\Gamma(n+\beta+1)} \simeq \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\, n^{\alpha+1}$$

then for $0 \leq \varphi \leq 1/n$,

$$|N_n(\varphi)| = O(n^{2\alpha+2}). \tag{3.9}$$

Proof. Using the order estimate (3.1) we obtain in the range $0 \leq \varphi \leq 1/n$,

$$
\begin{aligned}
M_{n-k}(\varphi) &= \frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} \lambda_m \, P_m^{(\alpha+1,\beta)}(\cos \varphi) \\
&= O\left[\frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} m^{2\alpha+2}\right] \\
&= O\,[(n-k)^{2\alpha+2}]\,.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
|N_n(\varphi)| &= \left|\frac{2^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k \, M_{n-k}(\varphi)\right| \\
&= O\left[\frac{n^{2\alpha+2}}{P_n} \sum_{k=0}^{n} p_k\right] \\
&= O\,(n^{2\alpha+2})\,.
\end{aligned}
$$

LEMMA 5. *For* $1/n \leq \varphi \leq \pi - 1/n$,

$$
\begin{aligned}
(3.10) \qquad |N_n(\varphi)| &= O\,(n^{\alpha-1/2}) \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{-\beta-1/2} + \\
&+ O\,(n^{\alpha-1/2}) \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{-\beta-3/2}.
\end{aligned}
$$

Proof. From Lemma 2, we have

$$
\begin{aligned}
M_{n-k}(\varphi) &= \frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} \lambda_m \, P_m^{(\alpha+1,\beta)}(\cos \varphi) \\
&= \frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)} m^{\alpha+1} \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{-\alpha-3/2} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{-\beta-1/2} \times \\
&\times \frac{m^{-1/2}}{\sqrt{\pi}} \left[\operatorname{Cos} \left\{\left(m + \frac{\alpha+\beta}{2} + 1\right) \varphi - \left(\alpha + \frac{3}{2}\right) \frac{\pi}{2}\right\} + \frac{O(1)}{m \operatorname{Sin} \varphi}\right] \\
&= \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)\sqrt{\pi}} \left[\frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} m^{\alpha+1/2} \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{-\alpha-3/2} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{-\beta-1/2} \times \right. \\
&\times \operatorname{Cos}\{m\varphi + \rho\varphi - \gamma\} + O\left(\frac{1}{n+1-k}\right) \sum_{m=0}^{m=n-k} m^{\alpha-1/2} \times \\
&\left. \times \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{-\beta-3/2}\right] \\
&= \Sigma_1 + \Sigma_2, \text{ say, where } \rho = \frac{\alpha+\beta}{2} + 1\,, \quad \gamma = \left(\alpha + \frac{3}{2}\right) \frac{\pi}{2}\,.
\end{aligned}
$$

Now

$$|\Sigma_2| = O\left[\frac{1}{(n+1-k)}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-3/2}\sum_{m=0}^{m=n-k} m^{\alpha-1/2}\right]$$

$$= O\left[(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-3/2}\right].$$

Also

$$\Sigma_1 = \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)\sqrt{\pi}}\,\frac{1}{(n+1-k)}\sum_{m=0}^{m=(n-k)} m^{\alpha+1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-3/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}\times$$

$$\times \operatorname{Cos}\{m\varphi+\rho\varphi-\gamma\}$$

$$= \frac{2^{-\alpha-\beta-1}}{\Gamma(\alpha+1)}\,\frac{1}{\sqrt{\pi}\,(n+1-k)}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-3/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}\Bigg[\operatorname{Cos}(\gamma-\rho\varphi)\times$$

$$\times\sum_{m=0}^{m=n-k} m^{\alpha+1/2}\operatorname{Cos} m\varphi + \operatorname{Sin}(\gamma-\rho\varphi)\sum_{m=0}^{m=n-k} m^{\alpha+1/2}\operatorname{Sin} m\varphi\Bigg].$$

Again using Abel's transformation

$$\sum_{m=0}^{m=n-k} m^{\alpha+1/2}\operatorname{Cos} m\varphi = O\,(n-k)^{\alpha+1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-1}, \qquad n\varphi \geq 1$$

and

$$\sum_{m=0}^{m=n-k} m^{\alpha+1/2}\operatorname{Sin} m\varphi = O\,(n-k)^{\alpha+1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-1}, \qquad n\varphi \geq 1\,.$$

Thus we have

$$|\Sigma_1| = O\left\{(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}\right\}.$$

Therefore

$$|M_{n-k}(\varphi)| = O\left\{(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}\right\}+$$

$$+ O\left\{(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-3/2}\right\}$$

and

$$|N_n(\varphi)| = O\left[\frac{1}{P_n}\sum_{k=0}^{k=n} p_k\,(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-3/2}\right]+$$

$$+ O\left[\frac{1}{P_n}\sum_{k=0}^{n} p_k\,(n-k)^{\alpha-1/2}\left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2}\left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}\right].$$

Now since

$$\sum_{k=0}^{n} p_k (n-k)^{\alpha-1/2} = \sum_{k=0}^{k=[n/2]} p_k (n-k)^{\alpha-1/2} + \sum_{k=[n/2]+1}^{k=n} p_k (n-k)^{\alpha-1/2}$$

$$= O(n^{\alpha-1/2}) \sum_{k=0}^{[n/2]} p_k + O(p_{[n/2]}) \left[(n-k)^{\alpha+1/2}\right]_{[n/2]+1}^{n}$$

$$= O(n^{\alpha-1/2}) P_n + O(np_n) n^{\alpha-1/2}$$

$$= O(n^{\alpha-1/2} P_n).$$

Consequently,

$$|N_n(\varphi)| = O(n^{\alpha-1/2}) \left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-3/2} +$$

$$+ O(n^{\alpha-1/2}) \left(\operatorname{Sin}\frac{\varphi}{2}\right)^{-\alpha-5/2} \left(\operatorname{Cos}\frac{\varphi}{2}\right)^{-\beta-1/2}.$$

LEMMA 5. *For* $\pi - 1/n \leq \varphi \leq \pi$,

$$|N_n(\varphi)| = O(n^{\alpha+\beta+1}). \tag{3.11}$$

Proof. On substituting $\varphi = \pi - t$, we have

$$M_{n-k}(\varphi) = \frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} \lambda_m P_m^{(\alpha+1,\beta)}(\cos\varphi)$$

$$= \frac{1}{(n+1-k)} \sum_{m=0}^{m=(n-k)} \lambda_m (-1)^m P_m^{(\beta,\alpha+1)}(\cos t), \text{ where } 0 < t \leq 1/n.$$

Hence using (3.1)

$$|M_{n-k}(\varphi)| = O\left[\frac{1}{(n+1-k)} \sum_{m=0}^{m=n-k} m^{\alpha+\beta+1}\right]$$

$$= O(n-k)^{\alpha+\beta+1},$$

and, therefore,

$$|N_n(\varphi)| = O\left[\frac{n^{\alpha+\beta+1}}{P_n} \sum_{k=0}^{n} p_k\right], \qquad \text{since } \alpha+\beta > -1$$

$$= O(n^{\alpha+\beta+1}).$$

4. *Proof of theorem.*

From (2.2), we have

$$\sigma_n(1) - A = \frac{1}{(n+1)} 2^{\alpha+\beta+1} \sum_{m=0}^{m=n} \lambda_m \times$$

$$\times \int_0^{\pi} \left(\operatorname{Sin}\frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos}\frac{\varphi}{2}\right)^{2\beta+1} \psi(\varphi) P_m^{(\alpha+1,\beta)}(\cos\varphi)\, d\varphi.$$

The Cesàro–Nörlund mean of the series (2.1) at $x = +1$ is given by

$$T_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k \, \sigma_{n-k}(1) .$$

Or

$$T_n - A = \frac{1}{P_n} \sum_{k=0}^{n} p_k \left[\sigma_{n-k}(1) - A\right]$$

$$= \frac{1}{P_n} \sum_{k=0}^{n} p_k \frac{2^{\alpha+\beta+1}}{(n+1-k)} \sum_{m=0}^{m=n-k} \lambda_m \times$$

$$\times \int_0^{\pi} \psi(\varphi) \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} P_m^{(\alpha+1,\beta)}(\cos \varphi)\, d\varphi ,$$

$$= \int_0^{\pi} \psi(\varphi) \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} N_n(\varphi)\, d\varphi .$$

In order to prove the theorem, we have to show that

$$I(\varphi) = \int_0^{\pi} \psi(\varphi) \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} N_n(\varphi)\, d\varphi = o(1), \quad \text{as } n \to \infty .$$

Write

$$I(\varphi) = \int_0^{1/n} + \int_{1/n}^{\delta} + \int_{\delta}^{\pi-1/n} + \int_{\pi-1/n}^{\pi}$$

δ being a suitably chosen, sufficiently small constant.

(4.1) $$= I_1 + I_2 + I_3 + I_4 , \quad \text{say.}$$

In I_1, we use the result (3.9), and thus,

$$|I_1| = O(n^{2\alpha+2}) \int_0^{1/n} |\psi(\varphi)| \left(\operatorname{Sin} \frac{\varphi}{2}\right)^{2\alpha+1} \left(\operatorname{Cos} \frac{\varphi}{2}\right)^{2\beta+1} d\varphi$$

$$= O(n^{2\alpha+2}) \int_0^{1/n} |\psi(\varphi)|\, \varphi^{2\alpha+1}\, d\varphi$$

$$= O(n) \int_0^{1/n} |\psi(\varphi)|\, d\varphi$$

(4.2) $$= o(1), \qquad \text{from (2.8).}$$

Again, making use of the result (3.10), we obtain

$$|\mathrm{I}_2| = \mathrm{O}(n^{\alpha-1/2}) \int_{1/n}^{\delta} |\psi(\varphi)| \left(\mathrm{Sin}\,\frac{\varphi}{2}\right)^{\alpha-3/2} d\varphi$$

$$= \mathrm{O}(n^{\alpha-1/2}) \int_{1/n}^{\delta} |\psi(\varphi)|\, \varphi^{\alpha-3/2}\, d\varphi$$

$$= \mathrm{O}(n^{\alpha-1/2}) \left[\{o(\varphi)\}\, \varphi^{\alpha-3/2}\right]_{1/n}^{\delta} + \mathrm{O}(n^{\alpha-1/2}) \int_{1/n}^{\delta} \{o(\varphi)\}\, \varphi^{\alpha-5/2}\, d\varphi\,,$$

since δ is chosen sufficiently small

$$= o(n^{\alpha-1/2}) + o(1) + o(n^{\alpha-1/2}) \int_{1/n}^{\delta} \varphi^{\alpha-3/2}\, d\varphi$$

$$= o(1), \qquad \text{since} \quad \alpha < 1/2\,. \tag{4.3}$$

Coming to I_3, we have

$$|\mathrm{I}_3| = \mathrm{O}\left[\int_{\delta}^{\pi-1/n} n^{\alpha-1/2} |\psi(\varphi)| \left(\mathrm{Sin}\,\frac{\varphi}{2}\right)^{\alpha-3/2} \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{\beta+1/2} d\varphi\right] +$$

$$+ \mathrm{O}\left[\int_{\delta}^{\pi-1/n} n^{\alpha-1/2} |\psi(\varphi)| \left(\mathrm{Sin}\,\frac{\varphi}{2}\right)^{\alpha-3/2} \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{\beta-1/2} d\varphi\right]$$

$$= \mathrm{O}\left[\int_{\delta}^{\pi-1/n} n^{\alpha-1/2} |f(\cos\varphi) - \mathrm{A}| \left(\mathrm{Cos}\,\frac{\varphi}{2}\right) \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{\beta-1/2} d\varphi\right] +$$

$$+ \mathrm{O}\left[\int_{\delta}^{\pi-1/n} n^{\alpha-1/2} |f(\cos\varphi) - \mathrm{A}| \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{\beta-1/2} d\varphi\right]$$

$$= \mathrm{O}(n^{\alpha-1/2}), \qquad \text{from} \quad (3.5)$$

$$= o(1), \qquad \text{since} \quad \alpha < 1/2\,. \tag{4.4}$$

Finally, making use of the estimate (3.11)

$$|\mathrm{I}_4| = \left|\int_{\pi-1/n}^{\pi} \psi(\varphi) \left(\mathrm{Sin}\,\frac{\varphi}{2}\right)^{2\alpha+1} \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{2\beta+1} \mathrm{N}_n(\varphi)\, d\varphi\right|$$

$$= \mathrm{O}(n^{\alpha+\beta+1}) \int_{\pi-1/n}^{\pi} |f(\cos\varphi) - \mathrm{A}| \left(\mathrm{Sin}\,\frac{\varphi}{2}\right)^{2\alpha+1} \left(\mathrm{Cos}\,\frac{\varphi}{2}\right)^{2\beta+1} d\varphi$$

$$= O(n^{\alpha+\beta+1}) \int_0^{1/n} |f(-\cos\varphi) - A|\, \varphi^{\beta-1/2}\, \varphi^{\beta+3/2}\, d\varphi$$

$$= O(n^{\alpha-1/2}) \int_0^{1/n} \varphi^{\beta-1/2}\, |f(-\cos\varphi) - A|\, d\varphi$$

$$= o(n^{\alpha-1/2}), \qquad \text{from} \quad (3.6)$$

$$(4.5) \qquad = o(1), \qquad \text{since} \quad \alpha < 1/2 .$$

Combining (4.1), (4.2), (4.3), (4.4) and (4.5), we have $I(\varphi) = o(1)$. This completes the proof of the theorem.

5. Now since Theorem A and Theorem of this paper run on similar lines, it may be interesting to compare the scope of applicability of the conditions under which these theorems hold. We show below by means of examples, that for some special sequences $\{p_n\}$ the condition (2.8) of the present theorem is less stringent than the condition (2.4) of the Theorem A.

The condition (2.4) implies that

$$(A) \qquad \psi(\varphi) = \int_0^t |f(\cos\varphi) - A|\, d\varphi = o\left[\frac{p(1/t)}{P(1/t)}\right]$$

and from the condition (2.8) we have

$$(B) \qquad \psi(\varphi) = \int_0^t |f(\cos\varphi) - A|\, d\varphi = o(t) .$$

(*a*) In the case of harmonic summability $p_n = 1/(n+1)$ and so that $P_n \sim \log n$.

The condition (A) gives

$$\Psi(t) = o\left(\frac{t}{\log 1/t}\right),$$

while condition (B) gives

$$\Psi(t) = o(t) .$$

(*b*) In a case for which $p_n = \dfrac{\log(n+1)}{(n+1)}$ and so $P_n \sim \log^2(n+1)$.

The condition (A) gives

$$\Psi(t) = o\left(\frac{t}{\log 1/t}\right),$$

While condition (B) gives

$$\Psi(t) = o(t) .$$

I am grateful to Prof. D. P. Gupta for his kind advice during the preparation of this paper.

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