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On the strong solutions of the Navier—Stokes equations in three dimensional space. Nota I

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Analisi matematica. — *On the strong solutions of the Navier-Stokes equations in three dimensional space* (*). Nota I di GIOVANNI PROUSE, presentata (**) dal Corrisp. L. AMERIO.

RIASSUNTO. — Si enunciano un teorema di unicità ed esistenza in grande per la soluzione q.o. del problema misto di Cauchy-Dirichlet relativo alle equazioni di Navier-Stokes in uno spazio a tre dimensioni ed una proprietà di massimo per tale soluzione. Si dimostra un teorema ausiliario relativo alle disequazioni di Navier-Stokes.

1. – PRELIMINARIES

Let Ω be an open, bounded set $\subset \mathbf{R}^3$ of class C^2 with boundary Γ and Q the cylinder $\Omega \times [0, T]$. Consider the Navier-Stokes equations

$$(1.1) \qquad \begin{cases} \dfrac{\partial \vec{u}}{\partial t} - \mu \Delta \vec{u} + (\vec{u} \times \operatorname{grad}) \vec{u} + \operatorname{grad} p = \vec{f} \\ \operatorname{div}_x \vec{u} = \sum_{j=1}^{3} \dfrac{\partial u_j}{\partial x_j} = 0 , \end{cases}$$

to which we associate the initial boundary value problem defined by

$$(1.2) \qquad \vec{u}(x, 0) = \vec{z}(x) \qquad (x \in \Omega)$$

$$(1.3) \qquad \vec{u}(x, t) = 0 \qquad (x \in \Gamma, 0 < t \leq T);$$

with $\vec{z} \in H_0^1(\Omega)$, $\Delta \vec{z} \in L^2(\Omega)$, $\operatorname{div}_x \vec{z} = 0$. The aim of this and of the subsequent Note II is to prove the following two main theorems.

THEOREM 1 (existence and uniqueness theorem): *If* $\vec{f} \in L^2(Q)$ *and is a conservative force* ($\vec{f} = \operatorname{grad}_x \chi$), *there exists one, and only one, solution almost-everywhere in* Q *of* (1.1) *satisfying* (1.2), (1.3).

THEOREM 2 (maximum principle): *If* $\vec{u}(x, t)$ *is the solution defined by Theorem 1, then*

$$(1.4) \qquad \max_{(x,t) \in \overline{Q}} |\vec{u}(x, t)| = \max_{x \in \overline{\Omega}} |\vec{u}(x, 0)| .$$

Observe that, by the assumptions made, $\vec{v} \in H_0^1(\Omega)$, $\Delta \vec{v} \in L^2(\Omega) \Rightarrow \Rightarrow \vec{v} \in H^2(\Omega) \subset C^0(\overline{\Omega})$; hence $|\vec{z}|$ has in $\overline{\Omega}$ a maximum value which will be denoted by M.

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We recall that, as is well known, an existence theorem in the large for "strong" solutions of the non homogeneous Navier–Stokes equations in 3 dimensional space holds, provided the known term and the initial data are "small" (see Kieselev and Ladyzenskaja [1], Kaniel and Shinbrot [2], Lions [3], Prouse [4]); in the theorem we prove here we assume that the external forces are conservative, but no hypotheses are made on the "smallness" of the initial data.

Since the main tool we shall use to obtain the theorems is constituted by the Navier–Stokes inequalities, we shall in § 2 give some preliminary results regarding such inequalities, while the main theorems will be proved in § 3, contained in Note II.

We begin by giving the following definitions.

$\mathcal{N}(\Omega)$: set of vectors $\vec{v}(x) = \{v_1(x), v_2(x), v_3(x)\}$ indefinitely differentiable and with compact support in Ω and such that $\operatorname{div}_x \vec{v} = 0$.

N^0: closure of $\mathcal{N}(\Omega)$ in $L^2(\Omega)$; N^0 is a Hilbert space, setting

$$(\vec{u}, \vec{v})_{N^0} = \int_\Omega \sum_{j=1}^{3} u_j v_j \, d\Omega .$$

N^1: closure of $\mathcal{N}(\Omega)$ in $H^1(\Omega)$; N^1 is a Hilbert space, setting

$$(\vec{u}, \vec{v})_{N^1} = \int_\Omega \sum_{j,k=1}^{3} \frac{\partial u_j}{\partial x_k} \frac{\partial v_j}{\partial x_k} \, d\Omega ;$$

the embedding of N^1 in N^0 is obviously compact.

K_M: closed convex set of N^1 defined by

$$K_M = \{\vec{v}(x) \mid \vec{v} \in N^1, |\vec{v}| \leq M\} . \tag{1.5}$$

$\mathcal{N}(Q)$: set of vectors $\vec{v}(x, t)$ indefinitely differentiable and with compact support in Q and such that $\operatorname{div}_x \vec{v} = 0$

$N^0(Q)$: closure of $\mathcal{N}(Q)$ in $L^2(Q)$ ($N^0(Q) = L^2(0, T; N^0)$).

$N^*(Q)$: set of vectors $\vec{v} \in L^2(Q)$ such that $\vec{v} = \operatorname{grad}_x \varphi$ $\left(\text{i.e. } v_i = \frac{\partial \varphi}{\partial x_i}\right)$. It can be proved (Prodi [5]) that $N^0(Q)$ and $N^*(Q)$ are mutually orthogonal and that the direct sum of $N^0(Q)$ with $N^*(Q)$ coincides with $L^2(Q)$.

The function $\vec{u}(x, t)$ will be called a *solution a.e. in Q of the Navier–Stokes equations satisfying the boundary condition* (1.3) *if*:

I_A) $\vec{u}(t) \in L^2(0, T; N^1)$, $\vec{u}'(t) \in L^2(0, T; N^0)$, $\Delta\vec{u}(t) \in L^2(0, T; N^0)$;

II_A) $\vec{u}(t)$ *satisfies*, $\forall \vec{h}(t) \in L^2(0, T; N^0)$, *the equation*

$$\int_0^T (\vec{u}'(t) - \mu\Delta\vec{u}(t) + (\vec{u}(t) \times \operatorname{grad}) \vec{u}(t) - \vec{f}(t), \vec{h}(t))_{N^0} \, dt = 0 , \tag{1.6}$$

or the equivalent equation

$$(1.6') \qquad \int_Q \left(\frac{\partial \vec{u}}{\partial t} - \mu\Delta\vec{u} + (\vec{u}\times\operatorname{grad})\,\vec{u} - \vec{f}\right)\times\vec{h}\,dQ = 0$$

where the symbol $\times$ denotes the scalar product and, by I_A), $\frac{\partial \vec{u}}{\partial t}$, $\Delta\vec{u}$, $(\vec{u}\times\operatorname{grad})\,\vec{u} \in L^2(Q)$.

Observe that from (1.6′) it follows that $\frac{\partial \vec{u}}{\partial t} - \mu\Delta\vec{u} + (\vec{u}\times\operatorname{grad})\,\vec{u} - \vec{f}$ is orthogonal to $N^0(Q)$, i.e.

$$\frac{\partial \vec{u}}{\partial t} - \mu\Delta\vec{u} + (\vec{u}\times\operatorname{grad})\,\vec{u} - \vec{f} \in N^*(Q)\,.$$

Hence there exists a function $p(x, t)$ such that

$$\frac{\partial \vec{u}}{\partial t} - \mu\Delta\vec{u} + (\vec{u}\times\operatorname{grad})\,\vec{u} - \vec{f} = -\operatorname{grad} p$$

a.e. in Q and this justifies the definition of solution given above.

Observe, moreover, that if we assume, as in Theorems 1 and 2, that $\vec{f} = \operatorname{grad}_x \chi$ (i.e. $\vec{f} \in N^*(Q)$), then we can set in (1.6), (1.6′) $\vec{f} = 0$, since $(\vec{f}, \vec{h})_{N^0} = 0 \quad \forall \vec{h} \in N^0(Q)$.

We shall say that $\vec{v}(x, t)$ is a *solution relative to* K_M *a.e. in* Q *of the Navier–Stokes inequalities satisfying the boundary condition* (1.3) *if*:

I_B) $\vec{v}(t) \in L^2(0, T; N^1)$, $\vec{v}'(t) \in L^2(0, T; N^1)$, $\Delta\vec{v}(t) \in L^2(0, T; N^0)$, $\vec{v}(t) \in K_M \quad \forall t \in [0, T]$;

II_B) $\vec{v}(t)$ *satisfies the inequality*

$$(1.7) \qquad \int_0^T (\vec{v}'(t) - \mu\Delta\vec{v}(t) + (\vec{v}(t)\times\operatorname{grad})\,\vec{v}(t) - \vec{f}(t), \vec{v}(t) - \vec{l}(t))_{N^0}\,dt \le 0$$

$\forall \vec{l}(t) \in L^2(0, T; N^0_{K_M})$, where $N^0_{K_M}$ is the closure of K_M in N^0.

It is easily seen that condition II_B) is equivalent to the following:

II'_B) $\vec{v}(t)$ *satisfies a.e. in* $[0, T]$ *the inequality*

$$(1.8) \qquad (\vec{v}'(t) - \mu\Delta\vec{v}(t) + (\vec{v}(t)\times\operatorname{grad})\,\vec{v}(t) - \vec{f}(t), \vec{v}(t) - \vec{\varphi})_{N^0} \le 0$$

$\forall \vec{\varphi} \in N^0_{K_M}$.

As in (1.6), (1.6′), if $\vec{f} \in N^*(Q)$, we can set in (1.7), (1.8) $\vec{f} = 0$.

Let us now give an interpretation of the definition given above. Let $\bar{t}$ be any point $\in [0, T]$ in which (1.8) holds and denote by $\Omega'(\bar{t})$ the set of points $(x, \bar{t})$ $(x \in \Omega)$ in which $|\vec{v}(x, \bar{t})| = M$ and by $\Omega''(\bar{t})$ the set $\Omega - \Omega'(\bar{t})$; since $\vec{v}(\bar{t}) \in H^2(\Omega)$, $\Omega''(\bar{t})$ is obviously open. We choose the test function $\vec{\varphi}$

in such a way that $\vec{\varphi} = \vec{v}(\bar{t}) - \lambda\vec{\psi}$, with $\vec{\psi} \in \mathcal{N}(\Omega''(\bar{t}))$ and $|\lambda|$ so small that $\vec{\varphi} \in K_M$. It follows then from (1.8) that

$$\lambda\,(\vec{v}'(\bar{t}) - \mu\Delta\vec{v}(\bar{t}) + (\vec{v}(\bar{t})\times\text{grad})\,\vec{v}(\bar{t}) - \vec{f}(\bar{t})\,,\,\vec{\psi})_{N^0(\Omega''(\bar{t}))} \leq 0\,.$$

Consequently, changing sign to the constant λ and observing that $\mathcal{N}(\Omega''(\bar{t}))$ is dense in $N^0(\Omega''(\bar{t}))$,

$$(1.9)\qquad (\vec{v}'(\bar{t}) - \mu\Delta\vec{v}(\bar{t}) + (\vec{v}(\bar{t})\times\text{grad})\,\vec{v}(\bar{t}) - \vec{f}(\bar{t})\,,\,\vec{\psi})_{N^0(\Omega''(\bar{t}))} = 0 \qquad \forall\vec{\psi} \in N^0(\Omega''(\bar{t})).$$

Hence $\vec{v}(x\,,t)$ *is a solution of the Navier–Stokes equations a.e. in the set* $Q'' \subset Q$ *in which* $|\vec{v}| < M$.

2. – An auxiliary theorem

Let us prove the following existence and uniqueness theorem for the solution of the Navier–Stokes inequalities.

Assume that $\vec{f}(t) \in L^2(0\,,T\,;L^2)$, $\vec{f}'(t) \in L^2(0\,,T\,;L^2)$, $\vec{z} \in K_M$, $\Delta\vec{z} \in N^0$ *and that* Ω *is of class* C^2. *There exists then a.e. in* Q *one and only one solution of the Navier–Stokes inequalities relative to* K_M *satisfying the initial and boundary conditions* (1.2) *and* (1.3).

We prove, first of all, the existence of such a solution. In what follows we shall set, for simplicity,

$$(2.1)\qquad b(\vec{u}\,,\vec{v}\,,\vec{w}) = \int_\Omega \sum_{j,k=1}^{3} u_k \frac{\partial v_j}{\partial x_k} w_j \,d\Omega = \int_\Omega (\vec{u}\times\text{grad})\,\vec{v}\times\vec{w}\,d\Omega\,.$$

Let β be a penalization operator relative to K_M (see Lions [3], ch. 3, § 5.2) defined by

$$\beta(\vec{y}) = J(\vec{y} - P_{K_M}\vec{y})$$

where J is a duality operator from N^1 to its dual space $(N^1)'$ and P_{K_M} is the operator " projection on K_M ". It is well known that β is a monotone, hemicontinuous operator from $L^2(0\,,T\,;N^1)$ to $L^2(0\,,T\,;(N^1)')$. We assume moreover to have chosen J in such a way that

$$(2.2)\qquad \langle J(\vec{y})\,,\vec{y}\rangle = \|J(\vec{y})\|_{(N^1)'}\,\|\vec{y}\|_{N^1} = \|\vec{y}\|^2_{N^1} \qquad \forall\vec{y} \in N^1;$$

under the assumptions made, this is always possible (see Lions [3], ch. 2, § 2.2) (1).

(1) For example, we can set

$$J(\vec{y}) = -\sum_{j,k=1}^{3} \frac{\partial^2 y_j}{\partial x_k^2}\,.$$

Let $\{\vec{g}_j\}$ be a basis in $N^1 \cap H^2(\Omega)$; we can assume that

$$(\vec{g}_j, \vec{y})_{N^1} = \lambda_j (\vec{g}_j, \vec{y})_{N^0} \qquad \forall \vec{y} \in N^1, \qquad (\vec{g}_j, \vec{g}_k)_{N^0} = \delta_{jk}.$$

Let, moreover, $\vec{z}_n$ be a linear combination of $\vec{g}_1, \cdots \vec{g}_n$ such that

$$\lim_{n\to\infty} \vec{z}_n \underset{N^1}{=} \vec{z} \quad , \quad \lim_{n\to\infty} \Delta\vec{z}_n \underset{N^0}{=} \Delta\vec{z} \quad , \quad \vec{z}_n \in K_M .$$

Setting

$$\vec{v}_n(t) = \sum_{j=1}^{n} \alpha_{jn}(t) \vec{g}_j ,$$

denoting by $\sigma(\xi)$ a function $\in C^1([0, B))$ $(B > M)$ with $\sigma(\xi) = 0$ when $0 \leq \xi \leq M$, $\sigma'(\xi) \geq 0$, $\lim_{\xi \to B^-} \sigma(\xi) = +\infty$ and observing that, if $\vec{y} \in H^2(\Omega)$ then $\beta(\vec{y}) \in L^2(\Omega)$, we consider the system of ordinary differential equations

$$(2.3) \qquad (\vec{v}_n'(t) - \mu\Delta\vec{v}_n(t) + n\beta(\vec{v}_n(t)) + \vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|) - \vec{f}(t), \vec{g}_j)_{N^0} + $$
$$+ b(\vec{v}_n(t), \vec{v}_n(t), \vec{g}_j) = 0 \qquad (j = 1, \cdots, n)$$

with the initial conditions

$$(2.4) \qquad \vec{v}_n(0) = \vec{z}_n .$$

System (2.3) obviously admits, for $t > 0$ sufficiently small, a unique solution satisfying (2.4). In order to obtain some a priori estimates on $\vec{v}_n(t)$ (from which, in particular, will follow the existence of such a solution on the whole of $[0, T]$), we multiply (2.3) by $\alpha_{jn}(t)$, add and integrate over $[0, t]$ $(0 < t \leq T)$; we then obtain

$$(2.5) \qquad \frac{1}{2}\|\vec{v}_n(t)\|_{N^0}^2 - \frac{1}{2}\|\vec{z}_n\|_{N^0}^2 + \int_0^t \{\mu\|\vec{v}_n(\eta)\|_{N^1}^2 + b(\vec{v}_n(\eta), \vec{v}_n(\eta), \vec{v}_n(\eta)) + $$
$$+ n(\beta(\vec{v}_n(\eta)), \vec{v}_n(\eta))_{N^0} + (\vec{v}_n(\eta)\,\sigma(|\vec{v}_n(\eta)|), \vec{v}_n(\eta))_{N^0} - (\vec{f}(\eta), \vec{v}_n(\eta))_{N^0}\}\, d\eta = 0 .$$

Since $(\beta(\vec{v}_n), \vec{v}_n)_{N^0} \geq 0$, $(\vec{v}_n\,\sigma(|\vec{v}_n|), \vec{v}_n)_{N^0} \geq 0$ and $b(\vec{v}_n, \vec{v}_n, \vec{v}_n) = 0$ (as can easily be verified bearing in mind (2.1) and integrating by parts), from (2.5) it follows that

$$(2.6) \qquad \|\vec{v}_n(t)\|_{N^0} \leq M_1 \quad , \quad \int_0^T \|\vec{v}_n(t)\|_{N^1}^2\, dt \leq M_2 ,$$

with M_1, M_2 independent of n. It is therefore possible to select from the sequence $\{\vec{v}_n(t)\}$ a subsequence (again denoted by $\{\vec{v}_n(t)\}$) such that

$$\lim_{n\to\infty} \vec{v}_n(t) \underset{L^2(0,T;N^1)}{=} \vec{v}(t) \quad , \quad \lim_{n\to\infty} \vec{v}_n(t) \underset{L^\infty(0,T;N^0)}{=} \vec{v}(t) \tag{2.7}$$

respectively in the weak and weak* topologies.

It follows, on the other hand, from (2.5), (2.6) that

$$\int_0^T (\beta(\vec{v}_n(t)), \vec{v}_n(t))_{N^0}\, dt \leq \frac{M_3}{n}$$

and, consequently,

$$\lim_{n\to\infty} \int_0^T (\beta(\vec{v}_n(t)), \vec{v}_n(t))_{N^0}\, dt = 0 . \tag{2.8}$$

Bearing in mind that, $\forall \vec{y} \in N^1 \cap H^2(\Omega)$

$$(J(\vec{y} - P_{K_M}\vec{y}), P_{K_M}\vec{y})_{N^0} \geq 0 ,$$

we obtain, by (2.2),

$$\int_0^T (\beta(\vec{v}_n(t)), \vec{v}_n(t))_{N^0}\, dt = \int_0^T (J(\vec{v}_n - P_{K_M}\vec{v}_n(t)), \vec{v}_n(t))_{N^0}\, dt = \tag{2.10}$$

$$= \int_0^T (J(\vec{v}_n(t) - P_{K_M}\vec{v}_n(t)), \vec{v}_n(t) - P_{K_M}\vec{v}_n(t))_{N^0}\, dt +$$

$$+ \int_0^T (J(\vec{v}_n(t) - P_{K_M}\vec{v}_n(t)), P_{K_M}\vec{v}_n(t))_{N^0}\, dt \geq \int_0^T \| \vec{v}_n(t) - P_{K_M}\vec{v}_n(t) \|^2_{N^1}\, dt .$$

Hence

$$\lim_{n\to\infty} \| \beta(\vec{v}_n) \|_{L^2(0,T;(N^1)')} = \lim_{n\to\infty} \| \vec{v}_n - P_{K_M}\vec{v}_n \|_{L^2(0,T;N^1)} = 0 . \tag{2.11}$$

Let $\vec{\varphi}$ be an arbitrary element of $L^2(0, T; N^1)$; we have, by (2.7), (2.8), (2.11),

$$-\int_0^T (\beta(\vec{\varphi}(t)), \vec{v}(t) - \vec{\varphi}(t))_{N^0}\, dt = \tag{2.12}$$

$$= \lim_{n\to\infty} \int_0^T (\beta(\vec{v}_n(t)) - \beta(\vec{\varphi}(t)), \vec{v}_n(t) - \vec{\varphi}(t))_{N^0}\, dt \geq 0 .$$

Setting $\vec{\varphi} = \vec{v} + \lambda\vec{\psi}$, $\vec{\psi}(t) \in L^2(0, T; N^1)$, $\lambda > 0$, it follows from (2.12) that

$$\int_0^T (\beta(\vec{v}(t) + \lambda\vec{\psi}(t)), \vec{\psi}(t))_{N^0}\, dt \leq 0$$

and consequently, since β is hemicontinuous, letting $\lambda \to 0$,

$$\int_0^T (\beta(\vec{v}(t)), \vec{\psi}(t))_{N^0}\, dt \leq 0 \qquad \forall \vec{\psi}(t) \in L^2(0, T; N^1),$$

that is

(2.13) $$\beta(\vec{v}(t)) = 0 \qquad \text{a.e. in } [0, T].$$

The function $\vec{v}(t)$ belongs therefore to K_M a.e. in $[0, T]$.

Let us now prove that

(2.14) $$|\vec{v}_n(x, t)| < B.$$

Observe, in fact, that from (2.5), (2.6) it follows that

(2.15) $$\int_0^T |(\vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|), \vec{v}_n(t))_{N^0}|\, dt \leq M_4.$$

If, on the other hand $|\vec{v}_n(x, t)| \geq B$ on a set $Q' \subset Q$, with $m(Q') > 0$, then, denoting by M^* any positive constant,

(2.16) $$\int_0^T |(\vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|), \vec{v}_n(t))_{N^0}|\, dt = \int_0^T \sigma(|\vec{v}_n|)\, \vec{v}_n \times \vec{v}_n\, dQ =$$
$$= \int_Q \sigma(|\vec{v}_n|)\, |\vec{v}_n|^2\, dQ > B^2 M^* m(Q').$$

Since we can choose M^* in such a way that $B^2 M^* m(Q') > M_4$, relation (2.16) is in contrast with (2.15) and therefore (2.14) holds.

Let us now multiply (2.3) by $-\lambda_j \alpha_{jn}(t)$; we obtain, adding and bearing in mind that, by the definitions given, $\{\vec{g}_j\}$ and λ_j are the eigenfunctions and eigenvalues of the operator Δ,

(2.17) $$\frac{1}{2}\frac{d}{dt}\|\vec{v}_n(t)\|^2_{N^1} + \mu\|\Delta\vec{v}_n(t)\|^2_{N^0} - b(\vec{v}_n(t), \vec{v}_n(t), \Delta\vec{v}_n(t)) -$$
$$-(n\beta(\vec{v}_n(t)) + \vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|) - \vec{f}(t), \Delta\vec{v}_n(t))_{N^0} = 0.$$

Observe now that

(2.18) $$-(\beta(\vec{v}_n), \Delta\vec{v}_n)_{N^0} = \sum_{j=1}^{3} \left\langle \frac{\partial}{\partial x_j}\beta(\vec{v}_n), \frac{\partial \vec{v}_n}{\partial x_j} \right\rangle$$

and, by the monotonicity of β, setting $\vec{v}_n(x\,,t)=0$ when $x \notin \overline{\Omega}$,

$$(2.19)\qquad \int\limits_{\Omega} (\beta(\vec{v}_n(x+h\,,t)) - \beta(\vec{v}_n(x\,,t))) \times (\vec{v}_n(x+h\,,t) - \vec{v}_n(x\,,t))\, d\Omega \geq 0 .$$

Since β satisfies a Lipschitz condition (N^1 being a Hilbert space) it follows from (2.18), (2.19) that

$$(2.20)\qquad -(\beta(\vec{v}_n)\,,\Delta\vec{v}_n)_{N^0} \geq 0 .$$

Moreover, by the definition of the function σ,

$$(2.21)\qquad -(\vec{v}_n\,\sigma(|\vec{v}_n|)\,,\Delta\vec{v}_n)_{N^0} = \sum_{j=1}^{3}\left(\frac{\partial}{\partial x_j}(\vec{v}_n\,\sigma(|\vec{v}_n|))\,,\frac{\partial\vec{v}_n}{\partial x_j}\right)_{N^0} =$$

$$= \sum_{j=1}^{3}\left(\sigma(|\vec{v}_n|)\frac{\partial\vec{v}_n}{\partial x_j}\,,\frac{\partial\vec{v}_n}{\partial x_j}\right)_{N^0} + \sum_{j=1}^{3}\left(\sigma'(|\vec{v}_n|)\frac{\partial|\vec{v}_n|}{\partial x_j}\vec{v}_n\,,\frac{\partial\vec{v}_n}{\partial x_j}\right)_{N^0} \geq$$

$$\geq \sum_{j=1}^{3}\int\limits_{\Omega}\sigma'(|\vec{v}_n|)\frac{\partial|\vec{v}_n|}{\partial x_j}\vec{v}_n \times \frac{\partial\vec{v}_n}{\partial x_j}\,d\Omega =$$

$$= \sum_{j,k=1}^{3}\int\limits_{\Omega}\sigma'(|\vec{v}_n|)\,\frac{v_{nk}\dfrac{\partial v_{nk}}{\partial x_j}}{|\vec{v}_n|}\,v_{nk}\frac{\partial v_{nk}}{\partial x_j}\,d\Omega \geq 0 .$$

On the other hand, bearing in mind (2.14), we have

$$(2.22)\qquad |b(\vec{v}_n\,,\vec{v}_n\,,\Delta\vec{v}_n)| \leq \max|\vec{v}_n|\,\|\vec{v}_n\|_{N^1}\,\|\Delta\vec{v}_n\|_{N^0} \leq B\,\|\vec{v}_n\|_{N^1}\,\|\Delta\vec{v}_n\|_{N^0} .$$

Hence, by (2.17), (2.20), (2.21), (2.22),

$$(2.23)\qquad \frac{1}{2}\frac{d}{dt}\|\vec{v}_n(t)\|_{N^1}^2 + \mu\,\|\Delta\vec{v}_n(t)\|_{N^0}^2 \leq$$

$$\leq B\,\|\vec{v}_n(t)\|_{N^1}\,\|\Delta\vec{v}_n(t)\|_{N^0} + |(\vec{f}(t)\,,\Delta\vec{v}_n(t))_{N^0}|$$

from which it follows that

$$(2.24)\qquad \|\vec{v}_n(t)\|_{N^1} \leq M_5 \quad , \quad \int_0^T \|\Delta\vec{v}_n(t)\|_{N^0}^2\,dt \leq M_6$$

with $M_5\,,M_6$ independent of n. We can therefore assume that

$$(2.25)\qquad \lim_{n\to\infty}\Delta\vec{v}_n(t) \underset{L^2(0,T\,;N^0)}{=} \Delta\vec{v}(t) \quad , \quad \lim_{n\to\infty}\vec{v}_n(t) \underset{L^\infty(0,T\,;N^1)}{=} \vec{v}(t)$$

respectively in the weak and weak* topologies.

Let us now differentiate (2.3); we obtain

$$(2.26)\qquad (\vec{v}_n''(t) - \mu\Delta\vec{v}_n'(t) + n(\beta(\vec{v}_n(t)))' + (\vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|))' - \vec{f}'(t), \vec{g}_j)_{N^0} + \\ + b(\vec{v}_n'(t), \vec{v}_n(t), \vec{g}_j) + b(\vec{v}_n(t), \vec{v}_n'(t), \vec{g}_j) = 0 \qquad (j = 1, \cdots, n).$$

Multiplying (2.26) by $\alpha_{jn}'(t)$ and adding we have, since $b(\vec{u}, \vec{v}, \vec{w}) = -b(\vec{u}, \vec{w}, \vec{v})$,

$$(2.27)\qquad \frac{1}{2}\frac{d}{dt}\|\vec{v}_n'(t)\|_{N^0}^2 + \mu\|\vec{v}_n'(t)\|_{N^1}^2 + (n(\beta(\vec{v}_n(t)))' + (\vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|))' - \\ - \vec{f}'(t), \vec{v}_n'(t))_{N^0} - b(\vec{v}_n'(t), \vec{v}_n'(t), \vec{v}_n(t)) + b(\vec{v}_n(t), \vec{v}_n'(t), \vec{v}_n'(t)) = 0 .$$

Observe that from (2.3), written for $t = 0$, it follows, by the assumptions made, that $\|\vec{v}_n'(0)\|_{N^0} \leq M_7$; hence, by (2.14), (2.27) we obtain, following a procedure analogous to that used to obtain (2.24),

$$(2.28)\qquad \|\vec{v}_n'(t)\|_{N^0} \leq M_8 \quad , \quad \int_0^T \|\vec{v}_n'(t)\|_{N^1}^2\, dt \leq M_9$$

with M_8, M_9 independent of n. We can therefore assume that

$$(2.29)\qquad \lim_{n\to\infty} \vec{v}_n'(t) \underset{L^2(0,T;N^1)}{=} \vec{v}'(t) \quad , \quad \lim_{n\to\infty} \vec{v}_n'(t) \underset{L^\infty(0,T;N^0)}{=} \vec{v}'(t),$$

respectively in the weak and weak* topologies.

It is obvious, by (2.7), (2.13), (2.25), (2.29), that the function $\vec{v}(t)$ satisfies condition I_B). It remains to be proved that $\vec{v}(t)$ satisfies also condition II_B) (or II_B')).

Let $\vec{\varphi}$ be an element of N^1 such that $|\vec{\varphi}(x)| < M$, $\Delta\vec{\varphi} \in N^0$, with

$$(2.30)\qquad \vec{\varphi} = \sum_{j=1}^{\infty} \gamma_j \vec{g}_j .$$

Setting

$$(2.31)\qquad \vec{\varphi}_p = \sum_{j=1}^{p} \gamma_j \vec{g}_j ,$$

it is obvious, since the embedding of $H^2(\Omega)$ in $C^0(\overline{\Omega})$ is continuous, that, when $p \geq \bar{p}$ sufficiently large, $\vec{\varphi}_p \in K_M$. Assuming that $p \geq \bar{p}$ and setting $\rho_j = \gamma_j$ when $j \leq p$, $\rho_j = 0$ when $j > p$, let us multiply (2.3) by $\alpha_{jn}(t) - \rho_j$; taking $n \geq p$, we obtain

$$(2.32)\qquad (\vec{v}_n'(t) - \mu\Delta\vec{v}_n(t) + n\beta(\vec{v}_n(t)) + \vec{v}_n(t)\,\sigma(|\vec{v}_n(t)|) - \vec{f}(t), \vec{v}_n(t) - \vec{\varphi}_p)_{N^0} + \\ + b(\vec{v}_n(t), \vec{v}_n(t), \vec{v}_n(t) - \vec{\varphi}_p) = 0 .$$

Since $\vec{\varphi}_p \in K_M$, $\beta(\vec{\varphi}_p) = \sigma(|\vec{\varphi}_p|) = 0$ and

$$(\beta(\vec{v}_n), \vec{v}_n - \vec{\varphi}_p)_{N^0} = (\beta(\vec{v}_n) - \beta(\vec{\varphi}_p), \vec{v}_n - \vec{\varphi}_p)_{N^0} \geq 0$$

$$(\vec{v}_n \sigma(|\vec{v}_n|), \vec{v}_n - \vec{\varphi}_p)_{N^0} = (\vec{v}_n \sigma(|\vec{v}_n|) - \vec{\varphi}_p \sigma(|\vec{\varphi}_p|), \vec{v}_n - \vec{\varphi}_p)_{N^0} \geq 0 ;$$

consequently

$$(2.33) \qquad (\vec{v}_n'(t) - \mu\Delta\vec{v}_n(t) - \vec{f}(t), \vec{v}_n(t) - \vec{\varphi}_p)_{N^0} + b(\vec{v}_n(t), \vec{v}_n(t), \vec{v}_n(t) - \vec{\varphi}_p) \leq 0.$$

Letting $n \to \infty$, it follows therefore, by (2.7), (2.25), (2.29), that $\vec{v}(t)$ satisfies the relation

$$(2.34) \qquad (\vec{v}'(t) - \mu\Delta\vec{v}(t) - \vec{f}(t), \vec{v}(t) - \vec{\varphi}_p)_{N^0} + b(\vec{v}(t), \vec{v}(t), \vec{v}(t) - \vec{\varphi}_p) \leq 0$$

$\forall \vec{\varphi}_p$ given by (2.30), (2.31). Since the space of such functions is dense in $N^0_{K_M}$, (2.34) (which coincides with (1.8)) holds $\forall \vec{\varphi} \in N^0_{K_M}$ and the existence theorem is therefore proved.

We now show that the solution is unique, i.e. that $\vec{v}(t)$ is the only function satisfying conditions I_B), II_B), with $\vec{v}(0) = \vec{z}$.

Let, in fact $y(t)$ be a function satisfying I_B) and such that

$$(2.35) \qquad \vec{y}(0) = \vec{z}$$

$$(2.36) \qquad \int_0^T \{(\vec{y}'(t) - \mu\Delta\vec{y}(t) - \vec{f}(t), \vec{y}(t) - \vec{l}(t))_{N^0} + b(\vec{y}(t), \vec{y}(t), \vec{y}(t) - \vec{l}(t))\}\, dt \leq 0$$

$\forall \vec{l}(t) \in L^2(0, T; N^0_{K_M})$. Setting in (1.7) $\vec{l}(t) = \vec{y}(t)$, in (2.36) $\vec{l}(t) = \vec{v}(t)$ and adding, we obtain denoting by $\vec{w}(t)$ the difference $\vec{v}(t) - \vec{y}(t)$,

$$(2.37) \qquad \int_0^T \{(\vec{w}'(t) - \mu\Delta\vec{w}(t), \vec{w}(t))_{N^0} + b(\vec{v}(t), \vec{v}(t), \vec{w}(t)) - b(\vec{y}(t), \vec{y}(t), \vec{w}(t))\}\, dt \leq 0 .$$

Since $w(0) = 0$, from (2.37) follows, by the same procedure used for the Navier–Stokes equations (see Prodi [6]), that $\vec{w}(t) = 0$ in $[0, T]$. The theorem is therefore completely proved.

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