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Matrix Substitutions in Summability

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Analisi matematica. — *Matrix Substitutions in Summability.* Nota di DAVID F. DAWSON, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — È stata dimostrata da A. Brudno che se A è una matrice regolare esiste allora una matrice A* regolare normale tale che A e A* sono mutualmente consistenti per le successioni limitate. L'A. estende questi risultati ad altri spazi di successioni.

A. Brudno [2] showed that if A is a regular matrix, then there exists a normal regular matrix A^* such that A and A^* are mutually consistent for bounded sequences. A. M. Tropper [5] gave a simple proof of Brudno's result. A matrix which is lower-semi with no zero in the main diagonal is said to be a normal matrix. The advantage in working with a normal matrix in summability stems from the fact that a normal matrix has a unique two-sided inverse [3].

The purpose of the present paper is to determine some other situations in which the substitution of one matrix for another can be effected without disturbing certain mapping (or summability) properties. Our results deal with Cesàro summability and sequences of bounded variation, and include an extension of a theorem of R. P. Agnew [1].

Let (C, k) denote the Cesàro summability matrix of nonnegative integral order k, where $(C, O) = I$, the identity matrix. Let C denote the set of all convergent complex sequences and let B denote the set of all bounded complex sequences. We will be concerned with the following sequence spaces:

$$C^{(k)} = \{ z : (C, k) z \in C \},$$

$$B^{(k)} = \{ z : (C, k) z \in B \},$$

$$BV = \{ z : \Sigma | z_p - z_{p+1} | < \infty \},$$

$$| C^{(k)} | = \{ z : (C, k) z \in BV \}.$$

Sequences in BV are said to be of bounded variation or absolutely convergent, while sequences in $| C^{(k)} |$ are said to be absolutely Cesàro summable of order k.

DEFINITION. *The statement that matrices* A *and* A′ *are mutually consistent over a set* S *with respect to* BV *means that if* $x \in S$, *then* Ax *and* $A'x$ *are both defined and are absolutely convergent to the same limit or else neither converges absolutely.*

DEFINITION. *The statement that* A *and* A′ *are mutually consistent over* S *with respect to* $C^{(j)}$ *(or* $| C^{(j)} |$*) means that if* $x \in S$, *then* Ax *and* $A'x$ *are both*

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defined and $(C, j)(Ax)$ *and* $(C, j)(A'x)$ *converge (converge absolutely) to the same limit or else neither converges (converges absolutely).*

Tropper's proof [5, p. 672] of Brudno's theorem is actually a proof of the following theorem.

THEOREM 1 (Brudno–Tropper). *If* A *is a matrix such that* $\Sigma_{q=1}^{\infty} | a_{pq} | < \infty$, $p = 1, 2, 3, \cdots$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* B *with respect to* C.

COROLLARY. *If* Ax *is defined for every* $x \in B$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* B *with respect to* C.

Proof. The hypothesis of the corollary implies the hypothesis of Theorem 1.

We will need a corollary to the following theorem of R. G. Cooke [3].

THEOREM 2. (Cooke). *Matrix* A *has the property that* $A(C^{(k)}) \subset C$ *if and only if the following conditions hold*:

$$(1) \qquad \{ a_{pq} \}_{p=1}^{\infty} \in C, \qquad q = 1, 2, 3, \cdots,$$

$$(2) \qquad \left\{ \sum_{q=1}^{\infty} a_{pq} \right\}_{p=1}^{\infty} \in C,$$

$$(3) \qquad \sup_p \sum_{q=1}^{\infty} q^k \, | \Delta_2^k a_{pq} | < \infty,$$

$$(4) \qquad \{ q^k a_{pq} \}_{q=1}^{\infty} \in B, \qquad p = 1, 2, 3, \cdots,$$

where $\Delta_2 a_{pq} = a_{pq} - a_{p,q+1}$.

COROLLARY. *Matrix* A *has the property that* $A(C^{(k)}) \subset C^{(j)}$ *if and only if the following conditions hold*:

$$1) \qquad \{ a_{pq} \}_{p=1}^{\infty} \in C^{(j)}, \qquad q = 1, 2, 3, \cdots,$$

$$2) \qquad \left\{ \sum_{q=1}^{\infty} a_{pq} \right\}_{p=1}^{\infty} \in C^{(j)},$$

$$3) \qquad \sup_p \sum_{q=1}^{\infty} q^k \left| \Delta_2^k \binom{p+j-1}{j}^{-1} \sum_{s_{j-1}=1}^{s_j=p} \cdots \sum_{s_1=1}^{s_2} \sum_{t=1}^{s_1} a_{tq} \right| < \infty,$$

$$4) \qquad \left\{ q^k \binom{p+j-1}{j}^{-1} \sum_{s_{j-1}=1}^{s_j=p} \cdots \sum_{s_1=1}^{s_2} \sum_{t=1}^{s_1} a_{tp} \right\}_{q=1}^{\infty} \in B, \qquad p = 1, 2, 3, \cdots.$$

Proof. We note that $(C, j)A$ takes $C^{(k)}$ to C iff A takes $C^{(k)}$ to $C^{(j)}$. By the theorem, $(C, j)A$ takes $C^{(k)}$ to C iff $(C, j)A$ has properties corresponding to (1)–(4), but these corresponding properties are 1)–4), since the p, q–entry in the matrix $(C, j)A$ is

$$\binom{p+j-1}{j}^{-1} \sum_{s_{j-1}=1}^{s_j=p} \cdots \sum_{s_1=1}^{s_2} \sum_{t=1}^{s_1} a_{tq}.$$

The matrix manipulation $(C, j)(Ax) = [(C, j)A]x$ used here is easily verified when Ax is defined.

THEOREM 3. *If* Ax *is defined for every* $x \in B^{(k)}$, *then there exists a matrix* A' *such that* $Ax = A'[(C, k)\, x]$ *for every* $x \in B^{(k)}$.

Proof. We first note that $q^k a_{pq} \to 0$ as $q \to \infty$, $p = 1, 2, 3, \cdots$. To see this, we suppose that for some p, $\{q^k a_{pq}\}_{q=1}^{\infty}$ does not converge to 0. Let $z_{2n-1} = (2n-1)^k$, $z_{2n} = -(2n-1)^k$, $n = 1, 2, 3, \cdots$. Clearly $z \in B^{(k)}$, but $\Sigma_{q=1}^{\infty} a_{pq} z_q$ diverges, since $\{a_{pq} z_q\}_{q=1}^{\infty}$ is not a null sequence. This contradicts the hypothesis that Ax is defined for every $x \in B^{(k)}$.

Suppose $x \in B^{(k)}$. Using summation by parts, we note that

$$(*) \qquad \sum_{q=1}^{n} a_{pq} x_2 = \sum_{q=1}^{n-1} (\Delta_2^1 a_{pq})\, S_q^{(1)}(x) + a_{pn}\, S_n^{(1)}(x),$$

where $S_q^{(1)}(x) = \Sigma_{p=1}^{q} x_p$. By a proof similar to the proof of Theorem 3 on pp. 484–485 of [4], it can be shown that if $y \in B^{(k)}$, then $S_n^{(r)}(y) = O(n^k)$, $r = 0, 1, \cdots, k$, where $S_n^{(r)}(y) = \Sigma_{p=1}^{n} S_p^{(r-1)}(y)$. Hence from (*) we see that

$$\sum_{q=1}^{\infty} a_{pq} x_q = \sum_{q=1}^{\infty} (\Delta_2^1 a_{pq})\, S_q^{(1)}(x),$$

since $a_{pn} S_n^{(1)}(x) = a_{pn} O(n^k) = n^k a_{pn} O(1) = o(1)$ as $n \to \infty$. This process can be continued to show that

$$\sum_{q=1}^{\infty} a_{pq} x_q = \sum_{q=1}^{\infty} (\Delta_2^r a_{pq})\, S_q^{(r)}(x), \qquad r = 1, 2, \cdots, k.$$

Thus

$$\sum_{q=1}^{\infty} a_{pq} x_q = \sum_{q=1}^{\infty} \left[\binom{q+k-1}{k} \Delta_2^k a_{pq}\right] \frac{S_q^{(k)}(x)}{\binom{q+k-1}{k}}.$$

Hence if A' is the matrix whose p, q–entry is $\binom{q+k-1}{k} \Delta_2^k a_{pq}$, then $Ax = A'[(C, k)\, x]$.

COROLLARY 1. *If* $A(C^{(k)}) \subset C^{(j)}$ *and* $q^k a_{pq} \to 0$ *as* $q \to \infty$, $p = 1, 2, 3, \cdots$, *then there exists a matrix* A' *such that* Ax *and* $A'[(C, k)\, x]$ *are defined and equal for every* $x \in B^{(k)}$.

Proof. Suppose $x \in B^{(k)}$. Since $A(C^{(k)}) \subset C^{(j)}$, then 1)–4) of the corollary to Theorem 2 hold, and from 3) it is easy to show that

$$\sum_{q=1}^{\infty} \binom{q+k-1}{k} |\Delta_2^k a_{pq}| < \infty, \qquad p = 1, 2, 3, \cdots.$$

Thus the series

$$\sum_{q=1}^{\infty} \left[\binom{q+k-1}{k} \Delta_2^k a_{pq}\right] \frac{S_q^{(k)}(x)}{\binom{q+k-1}{k}}$$

converges, $p = 1, 2, 3, \cdots$, since $\left\{ S_q^{(k)}(x) \binom{q+k-1}{k}^{-1} \right\}_{q=1}^{\infty} \in B$. Consequently, if A' is the matrix with p, q–entry $\binom{q+k-1}{k} \Delta_2^k a_{pq}$, then $A'[(C, k)x]$ is defined for every $x \in B^{(k)}$, and from the proof of Theorem 3, $Ax = A'[(C, k)x]$ for every $x \in B^{(k)}$ since $q^k a_{pq} \to 0$ as $q \to \infty$, $p = 1, 2, 3, \cdots$. This completes the proof.

R. P. Agnew [1] showed that a simple sufficient condition for a regular matrix A to sum a divergent sequence of 0's and 1's is that $a_{pq} \to 0$ as p, $q \to \infty$. As an application of Theorem 3, we have the following extension of this result of Agnew.

COROLLARY 2. *If a regular matrix* A *sums every sequence in* $C^{(k)}$, $q^k a_{pq} \to 0$ *as* $q \to \infty$, $p = 1, 2, 3, \cdots$, *and* $q^k \Delta_2^k a_{pq} \to 0$ *as* $p, q \to \infty$, *then* A *sums a sequence in* $B^{(k)} - C^{(k)}$.

Proof. From Corollary 1, there exists a matrix A' such that Ax and $A'[(C, k)x]$ are defined and equal for all $x \in B^{(k)}$, and A' has p, q–entry $\binom{q+k-1}{k} \Delta_2^k a_{pq}$. Clearly A' is regular and satisfies Agnew's sufficient condition mentioned above. Thus there exists a divergent sequence v of 0's and 1's such that $A'v$ converges. Let $u = (C, k)^{-1} v$. Then $u \in B^{(k)} - C^{(k)}$, and $Au = A'[(C, k)\{(C, k)^{-1} v\}] = A'v \in C$.

THEOREM 4. *If* Ax *is defined for every* $x \in B^{(k)}$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* $B^{(k)}$ *with respect to* $C^{(j)}$.

Proof. By Theorem 3, there exists a matrix A' such that for every $x \in B^{(k)}$, $Ax = A'[(C, k)x]$. Hence if $x \in B^{(k)}$, then

$$(C, j)(Ax) = (C, j)\{A'[(C, k)x]\} = [(C, j)A'][(C, k)x],$$

and consequently $[(C, j)A']y$ is defined for every $y \in B$. Thus by the corollary to Theorem 1, there exists a normal matrix A'' such that $(C, j)A'$ and A'' are mutually consistent over B with respect to C. Let

$$A^* = (C, j)^{-1}[A''(C, k)]$$

and note that A^* is a normal matrix. If $x \in B^{(k)}$, then

$$(C, j)(A^* x) = (C, j)[\{(C, j)^{-1}[A''(C, k)]\}x] = A''[(C, k)x].$$

Hence A and A^* are mutually consistent over $B^{(k)}$ with respect to $C^{(j)}$. Again, as in the proof of the corollary to Theorem 2, the matrix manipulations used above are easily verified.

COROLLARY. *If* $A(C^{(k)}) \subset C^{(j)}$ *and* $q^k a_{pq} \to 0$ *as* $q \to \infty$, $p = 1, 2, 3, \cdots$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* $B^{(k)}$ *with respect to* $C^{(j)}$.

Proof. The hypothesis implies, as in the proof of Corollary 1 to Theorem 3, that Ax is defined for every $x \in B^{(k)}$. Thus the conclusion follows from Theorem 4.

THEOREM 5. *If* Ax *is defined for every* $x \in B$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* B *with respect to both* C *and* BV.

Proof. The proof is along the lines of Tropper's proof of Brudno's Theorem [5, p. 672]. Since Ax is defined for every $x \in B$, then $\Sigma_{q=1}^{\infty} | a_{pq} | < \infty$, $p = 1, 2, 3, \cdots$. Let $\{ \varepsilon_p \}$ be a positive term of a decreasing sequence such that $\Sigma_{p=1}^{\infty} \varepsilon_p < \infty$. Let $\{ n_p \}_{p=1}^{\infty}$ be an increasing sequence of positive integers such that $n_1 = 1$ and $\Sigma_{q=n_p}^{\infty} | a_{pq} | < \varepsilon_p$, $p = 2, 3, 4, \cdots$. Define a normal matrix A^* as follows:

$$
\begin{aligned}
a_{nn}^* &= n^{-2} && n = 1, 2, 3, \cdots, \\
a_{nk}^* &= 0 && k > n, \\
a_{nk}^* &= a_{pn} && n_p < n \leqq n_{p+1}, \quad p \geqq 1, \quad 1 \leqq k < n.
\end{aligned}
$$

Let $x \in B$ and M be such that $| x_p | < M$, $p \geqq 1$. Let $\sigma_p = \Sigma_{q=1}^{\infty} a_{pq} x_q$ and $\sigma_p^* = \Sigma_{q=1}^{\infty} a_{pq}^* x_q$. If $n_p < n < n_{p+1}$, then

$$
| \sigma_n^* - \sigma_{n+1}^* | = | x_n n^{-2} - x_{n+1} (n+1)^{-2} - a_{n+1,n}^* x_n | \leqq 2 M n^{-2} + M | a_{pn} | .
$$

If $p > 1$, then

$$
\Big| | \sigma_{p-1} - \sigma_p | - | \sigma_{n_p}^* - \sigma_{n_p+1}^* | \Big| \leqq \Big| \sum_{q=n_p}^{\infty} (a_{p-1,q} - a_{pq}) x_q + x_{n_p} n_p^{-2}
$$

$$
- x_{n_p+1} (n_p + 1)^{-2} - a_{pn_p} x_{n_p} \Big| \leqq 2 M \varepsilon_{p-1} + 2 M n_p^{-2} + M \varepsilon_p .
$$

Hence

$$
\sum_{p=n_2}^{\infty} | \sigma_p^* - \sigma_{p+1}^* | \leqq \sum_{p=1}^{\infty} | \sigma_p - \sigma_{p+1} | + 2 M \sum_{p=1}^{\infty} p^{-2} + 4 M \sum_{p=1}^{\infty} \varepsilon_p ,
$$

and

$$
\sum_{p=2}^{\infty} | \sigma_{p-1} - \sigma_p | \leqq \sum_{p=2}^{\infty} | \sigma_{n_p}^* - \sigma_{n_p+1}^* | + 2 M \sum_{p=2}^{\infty} n_p^{-2} + 3 M \sum_{p=2}^{\infty} \varepsilon_{p-1} .
$$

Therefore if $Ax \in BV$, then $A^* x \in BV$, and vice versa. It is easily seen (as in Tropper's proof) that $\lim_n \sigma_r = \lim_n \sigma_n^*$ if either limit exists.

COROLLARY. *If* Ax *is defined for every* $x \in B^{(k)}$, *then there exists a normal matrix* A^* *such that* A *and* A^* *are mutually consistent over* $B^{(k)}$ *with respect to both* $C^{(j)}$ *and* $| C^{(j)} |$.

Proof. Repeat the proof of Theorem 4, except that in the appropriate step, apply Theorem 5 instead of the corollary to Theorem 1.

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