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Global Results and Asymptotically Selfinvariant Sets

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RENDICONTI

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Classe di Scienze fisiche, matematiche e naturali

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Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Teoria degli insiemi. — *Global Results and Asymptotically Self-invariant Sets.* Nota di G. S. LADDE e S. LEELA, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori, applicando il principio di confronto, dimostrano due teoremi di carattere globale.

Questi teoremi sono usati per ottenere condizioni sufficienti per la stabilità e la parziale stabilità e criteri di limitatezza per insiemi antiinvarianti asintoticamente.

1. INTRODUCTION

There has been much work done regarding the classification of different kinds of invariant sets, namely, invariant sets, conditionally invariant sets, asymptotically invariant sets and conditionally asymptotically invariant sets etc. and the study of Lyapunov stability relative to these various types of invariant sets has provided a natural framework to discuss many weaker concepts of stability, that are useful in applications [3, 4, 5, 6, 7]. Most theorems on stability or asymptotic stability of any kind of invariant set (or other convenient set) are proved by dividing a neighborhood of the set into a number of suitable subsets and showing that the solutions have certain desired behavior with respect to these subsets. Thus, an efficient way of generalizing stability theorems is to prove some global results in terms of arbitrary sets and apply them to study the various problems of stability and boundedness [8, 9]. This approach enlarges the class of useful Lyapunov functions and offers a great deal of flexibility.

(*) Nella seduta del 10 marzo 1973.

Although the partial stability criteria of the invariant set has been discussed by many authors [1, 2, 3, 10, 11, 12], partial stability of other kinds of invariant sets has not been studied so far. In this paper, we obtain two general results of global character using a single Lyapunov function and the comparison principle. This naturally generalizes some results in [8, 9]. As applications of our global results, we obtain sufficient conditions for the stability and partial stability criteria of an asymptotically self-invariant set, which generalizes the work of Rouche and Peiffer [10].

We also note that it is possible to apply our results to several other cases such as the stability criteria of conditionally asymptotically invariant sets, partial restrictive stability of an invariant set, boundedness of asymptotically invariant sets etc.

2. Main Results

We shall consider the differential system

$$\frac{dx}{dt} = f(t, x) \quad , \quad x(t_0) = x_0 , \tag{2.1}$$

and the scalar differential equation

$$\frac{du}{dt} = g(t, u) \quad , \quad u(t_0) = u_0 , \tag{2.2}$$

where $f \in C[R^+ \times E, R^n]$, E being an open set in R^n and $g \in C[R^+ \times R, R]$. Let $x(t, t_0, x_0)$, $u(t, t_0, u_0)$ denote any solution of (2.1), (2.2) through (t_0, x_0) and (t_0, u_0) respectively.

Let Y be a subspace of R^n and let P be the projection operator from R^n onto Y, i.e. $P : R^n \to Y$. As usual, we denote by $\bar{H}$, ∂H, PH, the closure, the boundary and the projection of H onto Y, respectively, for any set $H \subset R^n$.

The following main result offers a general set of sufficient conditions for preventing the solutions of (2.1) that start in a given set $H \subset R^n$ from passing through any given part of the boundary ∂PH.

THEOREM 2.1. *Assume that*

(i) $V \in C[R^+ \times E, R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) *there exist sets* $I_1, I_2 \subset R^+$, $H \subset R^n$ *and* $G \subset Y$ *such that* $I_1 \cap I_2 \neq \varnothing$, $\bar{H} \subset E$, *and* $G \subset \partial PH$;

(iii) $a \in C[R^+, R]$ *and* $V(t, x) \geq a(t)$ *for* $(t, x) \in R^+ \times A$, *where* $A = [x \in E : Px \in G]$;

(iv) $x_0 \in H$, $t_0 \in I_1$ *and* $V(t_0, x_0) < a(t_0)$;

(v) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times B$ *where* $B = [x \in E : Px \in PH]$,

$$D^+ V(t, x) = \limsup_{h \to 0^+} \frac{1}{h} [V(t + h, x + hf(t, x)) - V(t, x)] \leq$$
$$\leq g(t, V(t, x));$$

(vi) *Any solution* $u(t, t_0, u_0)$ *of the scalar differential equation* (2.2) *satisfies the inequality* $u(t, t_0, u_0) < a(t)$, $t \geq t_0$, *provided* $u_0 < a(t_0)$ *and* $t_0 \in I_2$. *Then, there exists no* $t^* > t_0$, $t_0 \in I = I_1 \cap I_2$ *such that*

$$x(t, t_0, x_0) \in B \quad , \quad t, \in [t_0, t^*) \quad \text{and} \quad x(t^*, t_0, x_0) \in A. \tag{2.3}$$

Proof. Suppose that the assertion of Theorem 2.1 is false. Then there exists a $t^* > t_0$, $t_0 \in I = I_1 \cap I_2$ such that (2.3) holds. This implies that $x(t^*, t_0, x_0) \in A$. Consequently, by the assumption (iii), we get

$$V(t^*, x(t^*, t_0, x_0)) \geq a(t^*). \tag{2.4}$$

We set $u_0 = V(t_0, x_0)$, where $t_0 \in I$. Then, because of (2.3), we have $x(t, t_0, x_0) \in B$ for $t \in [t_0, t^*)$. As a result, the hypotheses (i) and (v) yield, using a known comparison result [3], the inequality

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0) \quad , \quad t \in [t_0, t^*), \tag{2.5}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2). We now let $x_0 \in H$. Then, by (iv) and the choice $t_0 \in I$, we have

$$V(t_0, x_0) < a(t_0).$$

Since $u_0 = V(t_0, x_0)$ and $t_0 \in I$, we have $u_0 < a(t_0)$. This implies, by (vi) that

$$r(t^*, t_0, u_0) < a(t^*). \tag{2.6}$$

It therefore readily follows from (2.4), (2.5) and (2.6) that

$$a(t^*) \leq V(t^*, x(t^*, t_0, x_0)) \leq r(t^*, t_0, u_0) < a(t^*).$$

This contradiction proves the theorem.

Suppose that the solutions that start in a given set are required to reach another given set in finite time and remain there for all future time. The next global result yields sufficient conditions for such a behavior of solutions of (2.1).

THEOREM 2.2. *Assume that*

(i) $V \in C[R^+ \times E, R]$ *and* $V(t, x)$ *is locally Lipschitzian in* x;

(ii) *there exist sets* $F \subset E$, $E_0 \subset E$, $I_1 \subset R^+$ *such that* $x_0 \in F$, $t_0 \in I_1$ *implies that* $x(t, t_0, x_0) \in C_0$ *for* $t \geq t_0$, *where* $C_0 = [x \in E : Px \in PE_0]$;

(iii) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times C_0$, $D^+ V(t, x) \leq \leq g(t, V(t, x))$;

(iv) *there exists a set* $D \subset E_0$ *such that* $\bar{D} \subset E_0$ *and* $V(t, x) \geq a(t)$ *for* $(t, x) \in R^+ \times D_0$, *where* $a \in C[R^+, R]$ *and* $D_0 = [x \in E : Px \in \in P(E_0 \setminus D)]$;

(v) *there exists a set* $I_2 \subset R^+$ *such that* $I_1 \cap I_2 \neq \varnothing$ *and a number* $T_0 = T_0(t_0, u_0) > 0$, $t_0 \in I_2$, $u_0 > 0$ *such that for any solution* $u(t, t_0, u_0)$ *of* (2.2), *the relation*

$$u(t, t_0, u_0) < a(t) \quad , \quad t \geq t_0 + T_0 \quad , \quad t_0 \in I_2,$$

holds.

Then, there exists a $T = T(t_0, x_0)$ such that $x_0 \in F$, $t_0 \in I = I_1 \cap I_2$ implies that

$$Px(t, t_0, x_0) \in PD, \qquad \text{for} \quad t \geq t_0 + T.$$

Proof. Let $x_0 \in F$ and $t_0 \in I$, so that by (ii),

$$x(t, t_0, x_0) \in C_0, \qquad \text{for} \quad t \geq t_0.$$

Set $u_0 = V(t_0, x_0)$, $t_0 \in I$ and $T = T(t_0, x_0) = T_0(t_0, V(t_0, x_0))$.

Then, because of (i) and (iii), we obtain

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0) \quad , \quad t \geq t_0. \tag{2.7}$$

Let $\{t_k\}$ be a sequence such that $t_k \geq t_0 + T$, $t_0 \in I$ and $t_k \to \infty$ as $k \to \infty$.

Suppose that, if possible,

$$x(t_k, t_0, x_0) \in D_0, \qquad \text{for} \quad t_k \geq t_0 + T.$$

Consequently, the assumption (iv) yields the inequality

$$V(t_k, x(t_k, t_0, x_0)) \geq a(t_k). \tag{2.8}$$

This, however leads to a contradiction

$$a(t_k) \leq V(t_k, x(t_k, t_0, x_0)) \leq r(t_k, t_0, u_0) < a(t_k),$$

because of the relation (2.7) and hypothesis (v) and thus, the proof is complete.

Remark 2.1. The two theorems that have been proved above include as special cases the main global results in [8, 9]. All that is required to verify this is to take $I_1 = I_2 = R^+$ and $Y = R^n$, so that projection operator P becomes the identity operator. Notice also that we have preferred to employ a single Lyapunov function instead of a vector Lyapunov function, as was done in [8]. For our purposes, a single Lyapunov function is quite sufficient. As will be seen below, it is possible to apply our results to cover several situations of stability and boundedness criteria including those which have not yet found their way into the literature such as the partial stability of an asymptotically self-variant set.

3. APPLICATIONS

Let us begin by giving applications of Theorem 2.1. Let M be a compact set in R^n and suppose that it is asymptotically self-invariant with respect to (2.1). For the definitions of an asymptotically self-invariant (ASI) set and its stability criteria, see [3, 4, 6]. For any set M, let

$$S(M, \rho) = [x \in R^n : d(x, M) < \rho],$$

where $d(x, M)$ is the usual distance of a point $x \in R^n$ from the set M. The following result gives sufficient conditions for the stability of the ASI set M.

THEOREM 3.1. *Suppose that*

(i) $V \in C[R^+ \times S(M, \rho) \setminus M, R]$, $V(t, x)$ *is locally Lipschitzian in* x *and* $V(t, x) \to -\infty$ *as* $d(x, M) \to 0$ *and* $t \to \infty$;

(ii) $b \in C[(0, \rho), R]$ *and for* $(t, x) \in R^+ \times S(M, \rho) \setminus M$, $V(t, x) \geq \geq b(d(x, M))$;

(iii) $g \in C[R^+ \times R, R]$ *and for* $(t, x) \in R^+ \times S(M, \rho) \setminus M$, $D^+ V(t, x) \leq \leq g(t, V(t, x))$;

(iv) *For every* $r \in (0, \rho)$, *there exists a* $\tau(r) > 0$ *such that any solution* $u(t, t_0, u_0)$ *of* (2.2) *satisfies* $u(t, t_0, u_0) < b(r)$, $t \geq t_0 \geq \tau(r)$, *provided* $u_0 < b(r)$. *Then the* ASI *set* M *with respect to* (2.1) *is uniformly stable.*

Proof. Given $\varepsilon_0 \in (0, \rho)$, there exist numbers $\delta(\varepsilon_0) > 0$ and $\tau(\varepsilon_0) > 0$ such that $x_0 \in S(M, \delta) \setminus M$, $t_0 \geq \tau(\varepsilon_0)$ implies $V(t_0, x_0) < b(\varepsilon_0)$, because of the assumption (i). We set $E = S(M, \rho) \setminus M$, $H = S(M, \varepsilon_0) \setminus M$, $G = \partial S(M, \varepsilon_0)$, $Y = R^n$, $a(t) = b(\varepsilon_0)$ and $I_1 = I_2 = [\tau(\varepsilon_0), \infty)$, so that all the hypotheses of Theorem 2.1 are verified. Hence the conclusion follows.

Remark 3.1. Suppose that the set M is self-invariant with respect to (2.1). If we assume, in Theorem 3.1, that $V(t, x) \to -\infty$ as $d(x, M) \to 0$ for each $t \in R^+$ and that $\tau(r) = 0$ for every $r \in (0, \rho)$, then, we can conclude that the invariant set M is equi-stable.

We recall that the stability of the ASI set M is also called the eventual stability of the set M [3].

Let us represent the system (2.1) in an equivalent form by splitting it, namely

$$y' = f_1(t, y, z) \quad , \quad y(t_0) = y_0 ,$$

$$z' = f_2(t, y, z) \quad , \quad z(t_0) = z_0 ,$$

where $x = (y, z)$, $f = (f_1, f_2)$ and the vectors y, z belong to R^k and R^{n-k} respectively. Assume that the set $\{0\}$ is ASI relative to the system (2.1).

Denote by $S_y(\rho) = [y \in R^k : \|y\| < \rho]$. Then we have

THEOREM 3.2. *Assume that*

(i) $V \in C[R^+ \times S_y(\rho) \setminus \{0\} \times R^{n-k}, R]$, $V(t, y, z)$ *is locally Lipschitzian in* y *and* z *and* $V(t, y, z) \to -\infty$ *as* $\|y\| + \|z\| \to 0$ *and* $t \to \infty$;

(ii) $b \in C[(0, \rho), R]$ *and* $V(t, y, z) \geq b(\|y\|)$, *for* $(t, y, z) \in R^+ \times S_y(\rho) \setminus \{0\} \times R^{n-k}$;

(iii) $g \in C[R^+ \times R, R]$ *and for* $(t, y, z) \in R^+ \times S_y(\rho) \setminus \{0\} \times R^{n-k}$, $D^+ V(t, y, z) = \limsup_{h \to 0^+} \frac{1}{h} [V(t+h, y+hf_1(t, y, z), z+hf_2(t, y, z)) - V(t, y, z)] \leq g(t, V(t, y, z))$;

(iv) *for every* $r \in (0, \rho)$, *there exists a* $\tau(r) > 0$ *such that any solution* $u(t, t_0, u_0)$ *of* (2.2) *satisfies* $u(t, t_0, u_0) < b(r)$ *for* $t \geq t_0 \geq \tau(r)$, *provided* $u_0 < b(r)$. *Then the* ASI *set* $X = 0$ *with respect to* (2.1) *is partially stable.*

Proof. Let $\varepsilon_1 \in (0, \rho)$ be given. Then, by (i), there exist numbers $\delta(\varepsilon_1) > 0$, $\tau(\varepsilon_1) > 0$ such that $\|y_0\| + \|z_0\| < \delta(\varepsilon_1)$, $t_0 \geq \tau(\varepsilon_1)$ implies $V(t_0, y_0, z_0) < b(\varepsilon_1)$. Thus, setting $E = S_y(\rho) \setminus \{0\} \times R^{n-k}$, $H = [(y, z) : \|y\| + \|z\| < \varepsilon_1] \setminus \{0\}$, $G = P\partial H = [y \in R^k : \|y\| = \varepsilon_1]$, $Y = R^k$, $a(t) = b(\varepsilon_1)$ and $I_1 = I_2 = [\tau(\varepsilon_1), \infty)$, we see that all the hypotheses of Theorem 2.1 are satisfied. Hence the partial stability of the ASI set $x = 0$ follows.

Remark 3.2. Although much work is done regarding the partial stability of the invariant set $x = 0$, [1, 2, 3, 10, 11, 12], the concept of partial stability of an asymptotically self-invariant set has not been considered so far.

Next, as an application of Theorem 2.2, let us consider the following

THEOREM 3.3. *Assume that the hypotheses of Theorem 3.2. hold. Suppose further that* (i) $b(s)$ *is non-decreasing in* s; *and* (ii) *for every* $r \in (0, \rho)$, *there exists a* $\tau = \tau(r) > 0$ *and* $T = T(r) > 0$ *such that every solution* $u(t, t_0, u_0)$ *of* (2.2) *satisfies* $u(t, t_0, u_0) < b(r)$, $t \geq t_0 + T$, $t_0 \geq \tau$. *Then the* ASI *set* M *with respect to* (2.1) *is uniformly asymptotically stable.*

Proof. Since, by Theorem 3.1, the ASI set M is uniformly stable, we have, for $\varepsilon_0 = \rho$, a $\delta_0 = \delta(\rho) > 0$ such that $x_0 \in S(M, \delta_0)$ implies that $x(t, t_0, x_0) \in S(M, \rho)$, $t \geq t_0$. By setting $F = S(M, \delta_0)$, $E_0 = E = S(M, \rho) \setminus M$, and $D = S(M, \varepsilon_1)$ for any $\varepsilon_1 \in (0, \rho)$ and by choosing $a(t) = b(\varepsilon_1)$ we see that the conditions (ii) and (iv) of Theorem 2.2 are verified. The other hypotheses of Theorem 2.2 may be verified as in Theorem 3.1. Hence, it follows, by Theorem 2.2, that if $x_0 \in S(M, \delta_0)$, then $x(t, t_0, x_0) \in S(M, \varepsilon_1)$, for $t \geq t_0 + T$, thus proving the asymptotic stability of the ASI set M.

It is clear from the foregoing discussion that it is possible to apply our global results (Theorems 2.1 and 2.2) to prove results concerning the boundedness of ASI sets, stability and partial stability criteria of conditionally asymptotically invariant sets [5], etc., by choosing the sets E, H, G, Y, F, D and the function $a(t)$ in a suitable way.

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