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Highly non-supersoluble CLT groups

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Teoria dei gruppi. — *Highly non-supersoluble CLT groups.* Nota di S. BASKARAN, presentata (*) dal Corrisp. G. ZAPPA.

RIASSUNTO. — Si dice CLT–gruppo un gruppo finito che possieda sottogruppi di ogni ordine possibile. I gruppi di ordine libero da quadrati e i p–gruppi finiti sono CLT–gruppi; inoltre ogni sottogruppo di Hall di un CLT–gruppo è un CLT–gruppo. Nella presente Nota vengono determinati tutti i CLT–gruppi in cui nessun sottogruppo proprio, ad eccezione dei sottogruppi di Hall, dei p–sottogruppi e dei sottogruppi di ordine libero da quadrati, sia un CLT–gruppo.

1. A CLT group (always finite in this note) is one which possesses subgroups of all possible orders. Unlike nilpotency, supersolubility, solubility, the property of being CLT is not inherited by subgroups. It is therefore interesting to determine the CLT subgroups of a CLT group but this difficult question is not treated as such in this note. On the other hand, it is known [1] that a supersoluble group is precisely a CLT group all of whose proper subgroups are also CLT. The purpose of this note is to characterize those CLT groups all of whose subgroups, save those which have to be CLT, are non–CLT. In this context, it may be noted that prime power subgroups and, as is easily seen, subgroups with order equal to a square-free integer are CLT. Again,

(*) *Hall subgroups of a* CLT *group are* CLT.

For, if S is a Hall subgroup of the CLT group G with index m in G and d is any divisor of $|S|$ then $(d, m) = 1$ and so the intersection of S with a subgroup of index d in G is a subgroup of index dm in G and hence of index d in S.

The observation (*) is of course immediate from P. Hall's extended Sylow Theorem for soluble groups also but the reasoning given above is elementary.

The proof of the theorem of this note depends heavily on the following result of the Author proved elsewhere [2].

(**) *For two primes p, q there exists a non*–CLT *group of order pq^2 if and only if p is an odd divisor of $q+1$. If in addition q is also odd then every non-abelian group of order pq^2 is non*–CLT.

2. THEOREM. *If* G *is any* CLT *group then* G *does not have any* CLT *subgroup other than Hall subgroups and subgroups with order equal to either a*

(*) Nella seduta del 10 febbraio 1973.

square-free integer or a power of a prime if and only if G *satisfies one of the following.*

(1) G *has no subgroup other than the above mentioned subgroups*; *or equivalently*, G *is of order either a power of a prime or product of a square-free integer with a prime.*

(2) $G \cong S_4$ *or* $A_4 \times Z_2$ *where* S_4 *and* A_4 *are respectively the symmetric and the alternating groups on* 4 *letters.*

(3) $G \cong L \times Z_q$ *where* q *is an odd prime and* L *is a semi-direct product of* $Z_q \times Z_q$ *with a group of order* m, *a square-free odd integer dividing* $q+1$, *induced by a monomorphism.*

Proof. 'only if' part. Noting that G satisfies (1) if $|G|$ is a power of 2, let $|G| = 2^r \prod_{i=1}^{k} p_i^{r_i}$, $r \geq 0$, p_i odd primes, $r_i \geq 1$ and $p_i < p_j$ whenever $i < j$.

CASE I. $r \geq 1$. Suppose $r_i \geq 2$ for some i, a subgroup of order $2p_i^2$ is CLT and so $r = 1$ and $r_i = 2$ for this i. Further, if $r_i = r_j = 2$ for two distinct i and j with $p_i < p_j$ then a subgroup of order $p_i^2 p_j$ is CLT, a contradiction. It follows that either G satisfies (1), or $r_i = 1$ for each i and $r \geq 3$. Assuming that the latter is satisfied it is seen from (**) and the restrictions on G that 3 is the only odd prime divisor of $|G|$ and so $|G| = 2^r 3$. Let if possible $r > 3$. Then G contains a subgroup M of order $2^{r-1} 3$ and a subgroup H of order 24 which must be non-CLT. Since $|H \cap M| \geq 12$ it follows that $H \subseteq M$ as H contains a subgroup of order 6. So if S is a subgroup of order 3 contained in H then $N_H(S)$, $N_M(S)$ and $N_G(S)$ are all of order 6 because no subgroup of order 12 is CLT. Hence $[M : N_M(S)] = 2^{r-2}$ and $[G : N_G(S)] = 2^{r-1}$ which means that both 2^{r-2} and 2^{r-1} are congruent to unity modulo 3, an impossibility. Thus $r = 3$ and $|G| = 24$. Certainly G is not supersoluble as otherwise every subgroup of G and so in particular a subgroup of order 12 would be CLT, a contradiction. Consequently $G \cong S_4$ or $A_4 \times Z_2$.

CASE II. $r = 0$. Now $|G|$ is odd. If $r_i \neq 2$ for some $i \neq k$ (p_k is the largest prime divisor of $|G|$.) then a subgroup of order $p_i^2 p_j$ is CLT for all $j > i$. Hence either G satisfies (1), or $r_i = 1$ for all $i \neq k$ and $r_k \geq 3$. In the latter case $|G| = mq^t$ where m is a square-free odd integer, q an odd prime not dividing m and $t \geq 3$. If now p is any prime divisor of m then a subgroup of order pq^2 is non-CLT and so by (**) p divides $q+1$. It follows that m divides $q+1$. Let if possible $t > 3$. Pick out a prime divisor p of m and consider a Hall subgroup K of order pq^t. Then by (*) K is CLT and so K contains a subgroup M of order pq^{t-1} and a subgroup H of order pq^3 which must be non-CLT. As, however, H contains a subgroup of order pq by virtue of $q \equiv -1 \pmod{p}$, $|H \cap M| \neq pq^2$ i.e. $H \subseteq M$. A contradiction arises as in Case I because of $q \not\equiv 1 \pmod{p}$. Thus $|G| = mq^3$ with $q+1$ divisible by m.

Now G has a normal Sylow q–subgroup and since G is not supersoluble G has no normal subgroup of order m. Let S_m be a Hall subgroup of order m and let $E = N_G(S_m)$. By P. Hall's extended Sylow Theorem for soluble groups it follows that the number of distinct conjugates of S_m in G cannot be q or q^3 and so $|E| = mq$. But then E contains a subgroup S of order q which is normal in E and also in any subgroup of order q^2 containing it. If $N_G(S) \neq G$ then it would be a CLT subgroup of order mq^2, as is easily seen, a contradiction. Thus S is normal in G. If now L is a subgroup of order mq^2 then $S \cap L = \{1\}$ as otherwise S would be normal in L leading (for a similar reason) to a contradiction. Consequently $LS = G$ also and G is a semi-direct product of S with L. But again the homomorphism of L into Aut S (the group of automorphisms of S) through which G is obtained can only be trivial as $(|L|, q-1) = 1$. Thus $G = L \times S$. Next, the unique subgroup F of order q^2 contained in L cannot obviously be cyclic since L is non–CLT. If now p is any prime divisor of m then F cannot normalize any subgroup of order p as G has no CLT subgroup of order pq^2. L is therefore a semi-direct product of F with a Hall subgroup A of order m where the homomorphism of A into Aut F through which L is obtained is a monomorphism. This shows that G satisfies (3).

'If' part. The conclusion is trivial if G satisfies (1). Let G satisfy (2). Then the intersection of A_4 with any subgroup of order 12 cannot be of order 6 and so A_4 is the only subgroup of order 12 for G, and it is non–CLT. Finally, let G satisfy (3). More precisely, let $G = L \times Q$ where $Q \cong Z_q$ and L is a semi-direct product of a group $F \cong Z_q \times Z_q$ with a group A of order m induced by a monomorphism ψ. Let p be any prime divisor of m. If S_p is a subgroup of order p contained in A then the semi-direct product of F with S_p obtained by restricting the homomorphism ψ to S_p is by (**) a non–CLT subgroup of order pq^2 contained in L since p divides $q+1$ and ψ is a monomorphism. As any two (Hall) subgroups of order pq^2 of L are conjugate to each other it follows that every subgroup of order pq^2 contained in L is non–CLT. Now, if W is any subgroup of G with order pq^2 then $|W \cap L| \geq pq$ which means that $W \subseteq L$ (as otherwise $W \cap L$ being a subgroup of order pq contained in L is contained in a Hall subgroup R of order pq^2 of L and so R is CLT, a contradiction to the fact just mentioned). Every subgroup of G having order pq^2 being therefore a subgroup of L is non–CLT. This being true for every prime divisor p of m the conclusion follows immediately from (*).

Remarks. (i) Viewing Aut $(Z_q \times Z_q)$ as GL $(2, q)$ we can assert that the semi-direct product in (3) of the theorem is constructible. For, GL $(2, q)$ has a subgroup of order m when m divides $q+1$. In fact, when m is further odd, a subgroup of GL $(2, q)$ with order m is cyclic so that the Hall subgroups of order m of G in (3) of the theorem are cyclic. For, if B is a subgroup of GL $(2, q)$ with order m then B has trivial intersection with the centre of GL $(2, q)$ and so B is isomorphic to a subgroup C of PGL $(2, q)$, which, being of odd order, must be contained in PSL $(2, q)$. It follows that C must be cyclic as can easily be deduced, for instance, from Hauptsatz 8.27, p. 213, of Endliche Gruppen I by B. Huppert (Springer, 1967).

(ii) Calling a CLT group *highly non-supersoluble* if it possesses definitely one subgroup which is non-CLT and, if more, all subgroups as non-CLT subgroups except those mentioned in the theorem, it can be said that there are not "many" highly non-supersoluble CLT groups.

REFERENCES

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