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Periodically Extendible Space-Time Manifolds

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Relatività. — *Periodically Extendible Space–Time Manifolds.* Nota di JOHN R. PORTER, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Dal fatto che nella relatività generale una varietà compatta di tipo temporale ha caratteristica di Eulero–Poincaré nulla, segue la non esistenza di varietà periodiche di tipo temporale che posseggano ipersuperficie compatte di tipo spaziale. Ciò vale in particolare per varietà di tipo temporale che posseggono campi di vettori di Killing.

Let M be a four-dimensional C^∞ differentiable manifold without boundary and g a pseudo–Riemannian metric on M of signature -2. The pair (M, g) will be called an Einstein–Lorentz manifold. The existence of g on M implies the existence of a line element field on M. If M is non-compact this always exists, but for M compact, it constitutes a topological restriction in that the Euler–Poincare characteristic of M must vanish. For compact Einstein–Lorentz manifolds, this fact has been exploited by Porter and Thompson [1] and [2]. This paper considers periodically extendible Einstein–Lorentz manifolds and compactifications of these for a certain subclass which does not in general have closed time-like curves which are objectional on physical grounds and which compact Einstein–Lorentz manifolds necessarily possess.

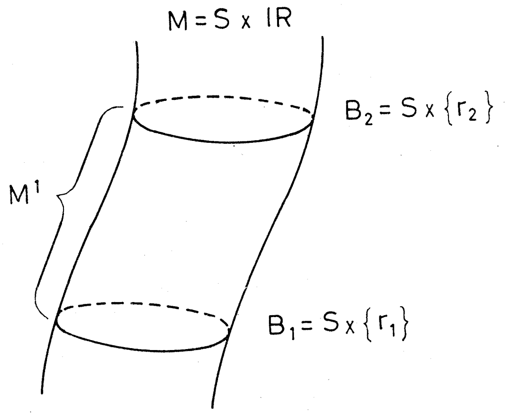

Fig. 1.

An Einstein–Lorentz manifold (M, g) is said to be *periodically extendible* if there exists an isometrically embedded Einstein–Lorentz manifold (M', g') with disjoint boundary components B_1 and B_2 ($\partial M = B_1 \cup B_2 \neq \varphi$ and $B_1 \cap B_2 = \varphi$) and a diffeomorphism φ between B_1 and B_2 which induces the identity on the first and second fundamental forms. A particularly interesting example would be obtained with $M = S \times \mathbf{R}$; $B_1 = S \times \{r_1\}$ and $B_2 = S \times \{r_2\}$, both space-like with respect to g and φ a diffeomorphism between B_1 and B_2 preserving the first and second fundamental forms, see fig. 1.

(*) Nella seduta del 10 febbraio 1973.

If (M, g) is a periodically extendible Einstein–Lorentz manifold with (M′, g') as before, then an Einstein–Lorentz manifold (M″, g'') is obtained from (M′, g') by using φ to identify points of B_1 with points of B_2. In the example, the resulting manifold could have the topology $S \times S^1$, but not necessarily. For example, there are two topologically distinct ways of obtaining a topological manifold from $S^1 \times I$ by identifying $S^1 \times \{0\}$ and $S^1 \times \{1\}$, namely the Torus and the Klein bottle, fig. 2.

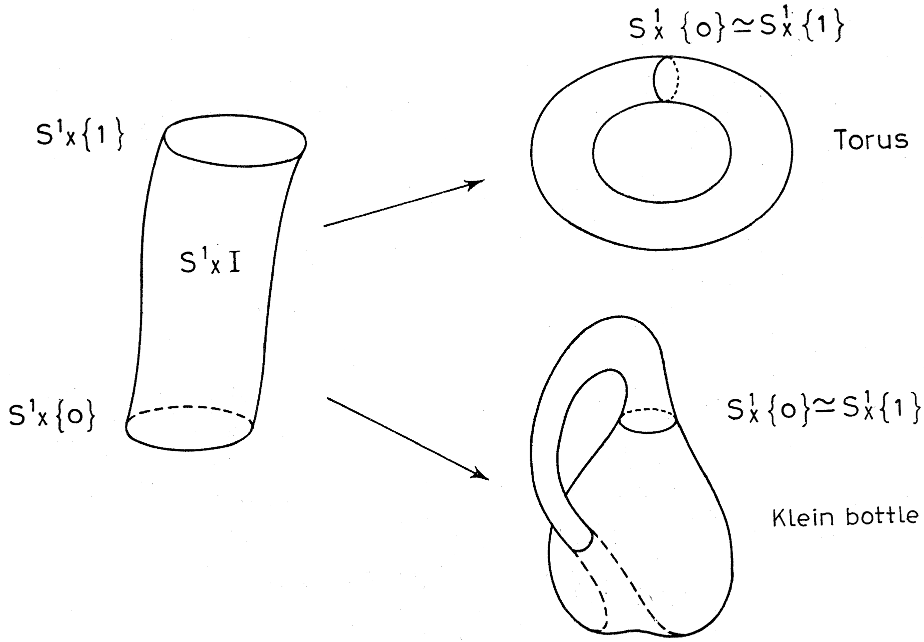

Fig. 2.

If (M′, g') is a compact Einstein–Lorentz manifold with boundary, then (M, g) is said to have *compact period*. In this case (M″, g'') will be compact and the results on compact Einstein–Lorentz manifolds from [1] and [2] may be employed. Among these results, the most interesting are:

THEOREM 1. *Compact Einstein–Lorentz manifolds of Pirani–Petrov types III*, N, *and* C *admit neither perfect fluids nor non-null electromagnetic fields.*

THEOREM 2. *Compact Einstein–Lorentz manifolds of Pirani–Petrov types II and* D *do not admit Robinson–Trautman vacuum solutions.*

Application of these theorems to the manifolds considered in this paper result in:

THEOREM 1′. *Periodically extendible Einstein–Lorentz manifolds with compact period of Pirani–Petrov types III*, N, *and* C *admit neither perfect fluids nor non-null electromagnetic fields.*

THEOREM 2′. *Periodically extendible Einstein–Lorentz manifolds with compact period of Pirani–Petrov types II and* D *do not admit Robinson–Trautman vacuum solutions.*

Note that the above theorems also apply to the following case. (M, g) possesses a compact hypersurface S without boundary and a Killing vector field transverse (nowhere tangent) to S. Then $M' = \{p \mid p \in M$ and $p = F(s, t)$ for $s \in S$ and $t \in [0, \delta]\}$ is compact where F is the flow of the Killing vector field and δ is less than or equal to the inf of the flow times from points on S. The flow then defines a diffeomorphism between S and $\{x \mid F(s, \delta)$ for $s \in S\}$ which preserves the first and second fundamental forms by virtue of the fact that it is the flow of a Killing vector field. In particular, if $M = S \times \mathbf{R}$ and S is a compact space-like hypersurface and M is stationary or static, then the non-existence of certain solutions is given by Theorems 1′ and 2′.

The objections against compact Einstein–Lorentz manifolds on the grounds that they necessarily possess closed time-like curves can be eliminated if such manifolds possess a space-like hypersurface without boundary. The manifold can be cut apart along the hypersurface, giving an Einstein–Lorentz manifold with boundary. Aleph–0 copies of this manifold can then be glued together along their boundaries to obtain a non-compact Einstein–Lorentz manifold which does not necessarily possess closed time-like curves.

References

[1] J. Porter and A. Thompson, *Some Topological Considerations in General Relativity*, « Rendiconti Accad. Naz. dei Lincei », *51*, 191–198 (1971).

[2] J. Porter and A. Thompson, *Robinson–Trautman Solutions on Compact Manifolds*, « Lettere Al Nuovo Cimento », *3*, 481–482 (1972).