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On the Betti number of a compact special almost Tachibana space

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Topologia. — *On the Betti number of a compact special almost Tachibana space.* Nota di SEIICHI YAMAGUCHI e FUMIZÔ MATSUMOTO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — Gli Autori provano che il primo numero del Betti di uno speciale spazio compatto M, quasi Tachibana, è zero, e che il terzo numero del Betti di M è zero o un numero pari.

§ 0. INTRODUCTION

Let M be a C^∞ n-dimensional almost Hermitian space with the structure tensor $(\varphi_j{}^i, g_{ji})$. M is called an almost Tachibana space (resp. a special almost Tachibana space) if the associated form is a Killing 2-form (resp. a special Killing 2-form with constant $\alpha (\neq 0)$). For such spaces, the following theorems have been proved.

THEOREM A [10]. *The first Betti number of a compact almost Tachibana space is zero or even.*

THEOREM B [5]. *In a compact almost Tachibana space, a necessary and sufficient condition that a pure p-form u be covariant almost analytic is that u is harmonic.*

THEOREM C [11]. *Any special almost Tachibana space is a 6-dimensional Einstein space.*

THEOREM D [11]. *Let* M *be a compact special almost Tachibana space with non-negative holomorphic bisectional curvature. Then* M *is isometric with a 6-dimensional sphere* S^6.

The purpose of this paper is to study harmonic 3-forms in a compact special almost Tachibana space and to prove the followings:

THEOREM 1. *In a compact special almost Tachibana space, if u is a harmonic 3-form, then* Φu *is also harmonic, where the operator* Φ *is given by* (1.8).

THEOREM 2. *The odd Betti number of a compact special almost Tachibana space is zero or even.*

§ 1. ALMOST TACHIBANA SPACE

Let M be an n-dimensional almost Hermitian space with local coordinate system $\{x^i\}$, almost complex structure $\varphi_j{}^i$ and metric tensor g_{ji}. To the almost Hermitian structure, a differential form $\varphi = 1/2\, \varphi_{ji}\, dx^j \wedge dx^i$ can be associated, where $\varphi_{ji} = \varphi_j{}^r g_{ri}$.

(*) Nella seduta del 10 febbraio 1973.

In this section, suppose that M is an almost Tachibana space. Then, by definition, it holds that

$$\nabla_j \varphi_{ih} + \nabla_i \varphi_{jh} = 0 , \tag{1.1}$$

where ∇ denotes the operator of covariant derivation with respect to the Riemannian connection. Therefore, to the almost Tachibana structure, a differential form

$$\psi = 1/3! \, \psi_{ijh} \, dx^i \wedge dx^j \wedge dx^h$$

can be associated, where we put

$$\psi_{ijh} = \nabla_i \varphi_{jh} .$$

Now, we shall recall some operators and identities between them in M [4]. The operators $\Gamma, \gamma, \mathfrak{D}$, etc. are defined by

$$(\Gamma u)_{i_0 \cdots i_p} = \sum_{\alpha=0}^{p} (-1)^\alpha \varphi_{i_\alpha}{}^r \nabla_r u_{i_0 \cdots \hat{i}_\alpha \cdots i_p} , \tag{1.2}$$

$$(\gamma u)_{i_0 \cdots i_p} = \sum_{\alpha \neq \beta} (-1)^\alpha \nabla_{i_\alpha} \varphi_{i_\beta}{}^r u_{i_0 \cdots \hat{i}_\alpha \cdots \overset{i_\beta}{\hat{r}} \cdots i_p} , \tag{1.3}$$

$$(\mathfrak{D} u)_{i_0 \cdots i_p} = \sum_{\alpha < \beta} (-1)^\alpha \varphi^{rs} \nabla_r \varphi_{i_\alpha i_\beta} u_{i_0 \cdots \hat{i}_\alpha \cdots \overset{i_\beta}{\hat{s}} \cdots i_p} , \tag{1.4}$$

$$(\mathbf{C} u)_{i_2 \cdots i_p} = \varphi^{rs} \nabla_r u_{s i_2 \cdots i_p} , \tag{1.5}$$

$$(c u)_{i_2 \cdots i_p} = \sum_{\alpha=2}^{p} \nabla^r \varphi^s{}_{i_\alpha} u_{r i_2 \cdots \overset{i_\alpha}{\hat{s}} \cdots i_p} , \tag{1.6}$$

$$(\vartheta u)_{i_2 \cdots i_p} = \sum_{\alpha=2}^{p} \varphi_{i_\alpha}{}^r \nabla^s \varphi_r{}^t u_{t i_2 \cdots \overset{i_\alpha}{\hat{s}} \cdots i_p} , \tag{1.7}$$

$$(\Phi u)_{i_1 \cdots i_p} = \sum_{\alpha=1}^{p} \varphi_{i_\alpha}{}^r u_{i_1 \cdots \overset{i_\alpha}{\hat{r}} \cdots i_p} , \tag{1.8}$$

$$\begin{aligned} (\mathrm{L} u)_{k h i_1 \cdots i_p} = {} & \varphi_{kh} u_{i_1 \cdots i_p} - \sum_{\alpha=1}^{p} \varphi_{i_\alpha h} u_{i_1 \cdots \overset{i_\alpha}{\hat{k}} \cdots i_p} \\ & - \sum_{\alpha=1}^{p} \varphi_{k i_\alpha} u_{i_1 \cdots \overset{i_\alpha}{\hat{h}} \cdots i_p} + \sum_{\alpha<\beta} \varphi_{i_\alpha i_\beta} u_{i_1 \cdots \overset{i_\alpha}{\hat{k}} \cdots \overset{i_\beta}{\hat{h}} \cdots i_p} , \end{aligned} \tag{1.9}$$

$$(\tilde{\mathbf{C}} u)_{i_3 \cdots i_p} = 1/2 \, \psi^{rst} \nabla_r u_{s t i_3 \cdots i_p} , \tag{1.10}$$

$$(\Lambda u)_{i_3 \cdots i_p} = 1/2 \, \varphi^{rs} u_{r s i_3 \cdots i_p} , \tag{1.11}$$

$$(\tilde{\Lambda} u)_{i_4 \cdots i_p} = 1/6 \, \psi^{rst} u_{r s t i_4 \cdots i_p} , \tag{1.12}$$

for any p-form $u = (u_{i_1 \cdots i_p})$, where we put $\psi^{rst} = \nabla^r \varphi^{st}$.

For the forms u_0, u_1 and u_2 of degree 0,1 and 2, we define $\tilde{\Lambda} u_2 = 0$, $\tilde{\Lambda} u_1 = \Lambda u_1 = \tilde{\mathbf{C}} u_1 = c u_1 = \vartheta u_1 = 0$ and $\tilde{\Lambda} u_0 = \Lambda u_0 = \tilde{\mathbf{C}} u_0 = c u_0 = \vartheta u_0 = \gamma u_0 = \mathfrak{D} u_0 = \mathbf{C} u_0 = \Phi u_0 = 0$.

Then the following relations hold good for any $p (\geq 0)$–form u,

$$(\Lambda L - L\Lambda)\, u = (m - p)\, u \,, \tag{1.13}$$

$$(d\Lambda - \Lambda d)\, u = (1/2\, c - \mathbf{C})\, u \,, \tag{1.14}$$

$$(\Gamma\Lambda - \Lambda\Gamma)\, u = (\delta + 1/2\, \vartheta)\, u \,, \tag{1.15}$$

$$(\mathbf{C}L - L\mathbf{C})\, u = (\mathfrak{D} - d)\, u \,, \tag{1.16}$$

$$(cL - Lc)\, u = -4\, \mathfrak{D} u \,, \tag{1.17}$$

$$(\delta L - L\delta)\, u = (\Gamma - 1/2\, \gamma)\, u \,, \tag{1.18}$$

$$(\gamma\Lambda - \Lambda\gamma)\, u = -2\, \vartheta u \,, \tag{1.19}$$

$$(\delta\tilde{\Lambda} + \tilde{\Lambda}\delta)\, u = 0 \,, \tag{1.20}$$

$$(\Delta\Lambda - \Lambda\Delta)\, u = -1/2\, (\delta c + c\delta)\, u - 3\, (\tilde{\Lambda} d + d\tilde{\Lambda})\, u \,, \tag{1.21}$$

$$(\Phi\Delta - \Delta\Phi)\, u = -1/2\, (\delta\gamma + \gamma\delta + cd + dc)\, u \,. \tag{1.22}$$

Let $(w\,,\,u)$ be the global inner product of p–forms w and u, then we have for a p–form u and a $(p+2)$–form v

$$(Lu\,,\,v) = (u\,,\,\Lambda v)\,. \tag{1.23}$$

§ 2. SPECIAL ALMOST TACHIBANA SPACE

In the following, we assume that M is a special almost Tachibana space. Then, the associated form φ is a special Killing 2–form with constant $\alpha\, (\neq 0)$, that is, it satisfies (1.1) and the equation

$$\nabla_k \nabla_j \varphi_{ih} = -\alpha\, (g_{kj}\, \varphi_{ih} - g_{ki}\, \varphi_{jh} + g_{kh}\, \varphi_{ji}). \tag{2.1}$$

Then we can easily show $\alpha > 0$. In M, the following identities are known [11]:

$$R_{jirs}\, \varphi^{rs} = -2\, \alpha \varphi_{ji} \,, \tag{2.2}$$

$$R_{ji} = 5\, \alpha g_{ji} \,, \tag{2.3}$$

$$R = 30\, \alpha \,. \tag{2.4}$$

Let us prove

PROPOSITION 2.1. *In* M, *we have for a p-form u*

$$(d\tilde{\Lambda} + \tilde{\Lambda} d)\, u = [\tilde{\mathbf{C}} - (p - 2)\, \alpha\Lambda]\, u \,. \tag{2.5}$$

Proof. Making use of (2.1), we have after some complicated calculations

$$
\begin{aligned}
6\,(\mathrm{d}\tilde{\Lambda}u + \tilde{\Lambda}\mathrm{d}u)_{i_3\cdots i_p} &= 3\,\psi^{rst}\,\nabla_r\,u_{sti_3\cdots i_p} - 3\,(p-2)\,\alpha\varphi^{rs}\,u_{rsi_3\cdots i_p} \\
&= 6\,[\tilde{\mathbf{C}}u - (p-2)\,\alpha\Lambda u]_{i_3\cdots i_p}\,.
\end{aligned}
$$

This completes the proof.

PROPOSITION 2.2. *In* M, *we have for a* p*-form* u

$$(\Delta\tilde{\Lambda} - \tilde{\Lambda}\Delta)\,u = (\delta\tilde{\mathbf{C}} - \tilde{\mathbf{C}}\delta)\,u + (p-2)\,\alpha\,(\Lambda\delta - \delta\Lambda)\,u\,. \tag{2.6}$$

Proof. Operating δ to (2.5), it follows that

$$\delta\mathrm{d}\tilde{\Lambda}u + \delta\tilde{\Lambda}\mathrm{d}u = \delta\tilde{\mathbf{C}}u - (p-2)\,\alpha\delta\Lambda u\,.$$

If we apply d to (1.20), then

$$\mathrm{d}\delta\tilde{\Lambda}u + \mathrm{d}\tilde{\Lambda}\delta u = 0\,.$$

Therefore, adding side by side of these identities and taking account of (1.20) and (2.5), we can easily find (2.6).

§ 3. PROOFS OF THEOREMS

In this section, we suppose that M is a compact special almost Tachibana space and u is a harmonic 3-form. We divide the proof of Theorem 1 into several Lemmas.

LEMMA 3.1. *In* M, *we have* $\tilde{\Lambda}u = 0$ *for* u.

Proof. For u, the equation (2.6) can be rewritten as

$$\Delta\tilde{\Lambda}u = \delta\tilde{\mathbf{C}}u - \alpha\delta\Lambda u\,. \tag{3.1}$$

On the other hand, making use of (2.2) and (2.4), it follows that

$$\delta\tilde{\mathbf{C}}u = -9\,\alpha\tilde{\Lambda}u\,. \tag{3.2}$$

Furthermore, we obtain

$$\delta\Lambda u = -3\,\tilde{\Lambda}u \tag{3.3}$$

by virtue of $\delta u = 0$. Thus, with the aid of (3.2) and (3.3), the equation (3.1) reduces to

$$\Delta\tilde{\Lambda}u = -6\,\alpha\tilde{\Lambda}u\,.$$

Consequently, the integral formula

$$(\Delta\tilde{\Lambda}u\,,\,\tilde{\Lambda}u) = -6\,\alpha\,(\tilde{\Lambda}u\,,\,\tilde{\Lambda}u)$$

is true and therefore we get $\tilde{\Lambda}u = 0$, because of $\alpha > 0$.

LEMMA 3.2. *In* M, *we have* $\Lambda u = \mathrm{L}u = 0$ *for* u.

Proof. In the first place, we shall show $\Lambda u = 0$. Applying ∇_i to $\psi_{jhk}\, u^{jhk} = 0$ obtained by Lemma 3.1 and taking account of (2.1), it holds that

$$\psi^{jhk}\, \nabla_i\, u_{jhk} = 3\, \alpha\varphi^{jh}\, u_{jhi}\,. \tag{3.4}$$

By the way, since the equation $(du)_{ijhk} = 0$ holds good for u, if we contract this with ψ^{jhk}, then

$$3\, \psi^{jhk}\, \nabla_j\, u_{hki} = \nabla_i\, u_{jhk}\, \psi^{ihk}\,, \tag{3.5}$$

and we get with the aid of (3.4) and (3.5)

$$\psi^{jhk}\, \nabla_j\, u_{hki} = \alpha\varphi^{jh}\, u_{jhi}\,,$$

or

$$\widetilde{\mathbf{C}}u = \alpha\Lambda u\,. \tag{3.6}$$

Hence, by making use of (1.21) and (2.1), it follows that

$$\Delta\Lambda u = -\, 2\, \alpha\Lambda u\,. \tag{3.7}$$

M being compact, it means that $\Lambda u = 0$. In the next place, let us prove $\mathrm{L}u = 0$. Regarding to $m = p = 3$ and $\Lambda u = 0$, we have $\Lambda \mathrm{L}u = 0$ from (1.13). Then the integral formula

$$0 = (\Lambda \mathrm{L}u\,, u) = (\mathrm{L}u\,, \mathrm{L}u)$$

holds good, which means that $\mathrm{L}u = 0$.

LEMMA 3.3. *In* M, *we have* $\mathbf{C}u = cu = 0$ *and* $\Gamma u = \gamma u = 0$ *for* u.

Proof. Regarding to Lemma 3.2, (1.16) and (1.17), we have

$$\mathrm{L}\mathbf{C}u = -\, \mathfrak{T}u \quad , \quad \mathrm{L}cu = 4\, \mathfrak{T}u \tag{3.8}$$

for u, and consequently, by virtue of (1.14) and (3.8), it holds that $\mathfrak{T}u = 0$. Moreover, it follows that

$$\Lambda\mathrm{L}\mathbf{C}u - \mathrm{L}\Lambda\mathbf{C}\, u = \mathbf{C}u$$

from (1.13), (3.8) and $\mathfrak{T}u = 0$, which denotes

$$\mathrm{L}\Lambda\mathbf{C}u = -\, \mathbf{C}u\,.$$

Hence, the integral formula

$$(\mathrm{L}\Lambda\mathbf{C}u\,, \mathbf{C}u) = -\, (\mathbf{C}u\,, \mathbf{C}u)$$

is true, which shows that $\mathbf{C}u = 0$.

Furthermore, by virtue of $\mathbf{C}u = 0$ and (1.14), we find $cu = 0$. Similarly, from Lemma 3.2, (1.13), (1.15), (1.18) and (1.19), we can obtain $\Gamma u = \gamma u = 0$. These complete the proof of Lemma 3.3.

Proof of Theorem 1. As the equations $\gamma u = 0$ and $cu = 0$ are true for u, we can easily obtain $\Delta \Phi u = 0$ from (1.22). This means that Φu is harmonic.

Proof of Theorem 2. The first Betti number of M is zero, because of Theorem C and $R > 0$. Moreover by making use of Theorem 1 and the method of [1] and [9], the third Betti number of M is zero or even. This completes the proof.

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