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On semi-continuous mappings

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Topologia. — *On semi-continuous mappings.* Nota di TAKASHI NOIRI, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Il presente lavoro espone alcune proprietà degli insiemi semiaperti e delle applicazioni semicontinue relative a spazi topologici, fra cui la seguente. Dato un insieme di applicazioni $f_\alpha : x_\alpha \to y_\alpha$, il loro prodotto $f, \Pi X_\alpha \to \Pi Y_\alpha$ [dove $f((x_\alpha)) = (f_\alpha(x_\alpha))$] risulta semicontinuo se, e soltanto se, tale è ciascuna delle f_α.

1. INTRODUCTION

In 1963, N. Levine [3] defined a subset A of a topological space X to be *semi-open* if there exists an open set U in X such that $U \subset A \subset \mathrm{Cl}\, U$, where $\mathrm{Cl}\, U$ denotes the closure of U in X. He also defined a mapping f of a topological space X into a topological space Y to be *semi-continuous* if for any open set V in Y, $f^{-1}(V)$ is a semi-open set in X. The purpose of the present note is to give a generalization of the following two theorems in [3] and to investigate some properties of semi-open sets and semi-continuous mappings.

THEOREM A. – *Let* X_1 *and* X_2 *be topological spaces. If* A_i *is a semi-open set in* X_i *for* $i = 1, 2$, *then* $A_1 \times A_2$ *is a semi-open set in the product space* $X_1 \times X_2$.

THEOREM B. – *Let* X_i *and* Y_i *be topological spaces and* $f_i : X_i \to Y_i$ *be a semi-continuous mapping for* $i = 1, 2$. *Then a mapping* $f : X_1 \times X_2 \to Y_1 \times Y_2$ *defined by putting* $f(x_1, x_2) = (f_1(x_1), f_2(x_2))$ *is semi-continuous.*

Throughout the present note X and Y will always denote topological spaces on which no separation axioms are assumed. No mapping $f : X \to Y$ is assumed to be continuous unless explicitly stated. When A is a subset of X, $\mathrm{Cl}\, A$ and $\mathrm{Int}\, A$ denote the closure of A and the interior of A respectively. Moreover, $\mathrm{Cl}_A B$ (resp. $\mathrm{Int}_A B$) denotes the closure (resp. interior) of a subset B of A with respect to the subspace A. We shall denote the family of all semi-open sets in X by SO(X).

2. SEMI-OPEN SETS

The intersection of two semi-open sets is not always semi-open [3, Remark 5]. However, we have the following lemma.

LEMMA 1. – *If* U *is open and* A *is semi-open, then* $U \cap A$ *is semi-open.*

Proof. Since A is semi-open, there exists an open set G such that $G \subset A \subset \mathrm{Cl}\, G$. It follows from the openness of U that $U \cap \mathrm{Cl}\, G \subset \mathrm{Cl}\,(U \cap G)$.

(*) Nella seduta del 10 febbraio 1973.

Hence we have $G \cap U \subset A \cap U \subset \mathrm{Cl}(G \cap U)$. This implies that $A \cap U$ is semi-open because $G \cap U$ is open.

Let A and X_0 be subsets of X such that $A \subset X_0$. Theorem 6 of [3] stated that if $A \in SO(X)$, then $A \in SO(X_0)$. The converse is false [3, Remark 2]. However, if $X_0 \in SO(X)$, then the converse is true, as shown by the following theorem.

THEOREM 1. – *Let* A *and* X_0 *be subsets of* X *such that* $A \subset X_0$ *and* $X_0 \in SO(X)$. *Then,* $A \in SO(X)$ *if and only if* $A \in SO(X_0)$.

Proof. The necessity is Theorem 6 of [3] itself; hence we need only prove the sufficiency. Suppose A is semi-open in X_0. Then there exists an open set U_0 in X_0 such that $U_0 \subset A \subset \mathrm{Cl}_{X_0} U_0$. Since U_0 is open in X_0, there exists an open set U in X such that $U_0 = U \cap X_0$. Therefore, we have $U \cap X_0 \subset A \subset \mathrm{Cl}_{X_0}(U \cap X_0) \subset \mathrm{Cl}(U \cap X_0)$. Since $X_0 \in SO(X)$ and U is open, by Lemma 1, $U \cap X_0$ is semi-open in X. Hence by Theorem 3 of [3], A is semi-open in X.

LEMMA 2. – A *is semi-open if and only if* $\mathrm{Cl}\,A = \mathrm{Cl}\,\mathrm{Int}\,A$.

Proof. *Necessity*. Suppose A is semi-open. Then by Theorem 1 of [3], we have $A \subset \mathrm{Cl}\,\mathrm{Int}\,A$ and so $\mathrm{Cl}\,A \subset \mathrm{Cl}\,\mathrm{Int}\,A$. On the other hand, we have $\mathrm{Int}\,A \subset A$ and hence $\mathrm{Cl}\,\mathrm{Int}\,A \subset \mathrm{Cl}\,A$. Consequently, we obtain $\mathrm{Cl}\,A = \mathrm{Cl}\,\mathrm{Int}\,A$.

Sufficiency. By the hypothesis, we have $\mathrm{Int}\,A \subset A \subset \mathrm{Cl}\,A = \mathrm{Cl}(\mathrm{Int}\,A)$. Hence A is semi-open.

LEMMA 3 (Kawashima [2]). – *Let* $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ *be any family of topological spaces and* ΠA_α *a subset of* ΠX_α, *where* ΠX_α *denotes the product space. Then,*

(1) $\mathrm{Int}\,\Pi A_\alpha = \Pi\,\mathrm{Int}\,A_\alpha$ *if* $A_\alpha = X_\alpha$ *except for a finite number of* $\alpha \in \mathcal{A}$ *and* $\Pi\,\mathrm{Int}\,A_\alpha \neq \varnothing$,

(2) $\mathrm{Cl}\,\Pi A_\alpha = \Pi\,\mathrm{Cl} A_\alpha$.

Proof. For the statement (1), see Lemma 2 of [2]. The statement (2) is well known.

LEMMA 4. – *If* A *is a non-empty semi-open set, then* $\mathrm{Int}\,A \neq \varnothing$.

Proof. Since A is semi-open, by Lemma 2, we have $\mathrm{Cl}\,A = \mathrm{Cl}\,\mathrm{Int}\,A$. Suppose $\mathrm{Int}\,A$ is empty. Then we have $\mathrm{Cl}\,A = \varnothing$ and hence $A = \varnothing$. This is contrary to the hypothesis. Therefore, $\mathrm{Int}\,A$ is not empty.

The following theorem is a generalization of Theorem A.

THEOREM 2. – *Let* $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ *be any family of topological spaces,* $X = \Pi X_\alpha$ *the product space and* $A = \prod_{j=1}^{n} A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$ *a non-empty subset of* X, *where* n *is a positive integer. Then,* $A_{\alpha_j} \in SO(X_{\alpha_j})$ *for each* $j\ (1 \leqq j \leqq n)$ *if and only if* $A \in SO(X)$.

Proof. Necessity. Suppose $A_{\alpha_j} \in SO(X_{\alpha_j})$ for each j $(1 \leqq j \leqq n)$. Since A is non-empty, $A_{\alpha_j} \neq \emptyset$ for each j $(1 \leqq j \leqq n)$. Moreover, $A_{\alpha_j} \in SO(X_{\alpha_j})$ and hence by Lemma 4, $\mathrm{Int}\, A_{\alpha_j}$ is non-empty. Thus $\prod_{j=1}^{n} \mathrm{Int}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$ is non-empty. Hence by Lemma 2 and 3, we have $\mathrm{Cl}\,\mathrm{Int}\, A = \prod_{j=1}^{n} \mathrm{Cl}\,\mathrm{Int}\, A_{\alpha_j} \times \times \prod_{\alpha \neq \alpha_j} X_\alpha = \prod_{j=1}^{n} \mathrm{Cl}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha = \mathrm{Cl}\, A$ because A_{α_j} is semi-open for each j $(1 \leqq j \leqq n)$. Again by Lemma 2, we have $A \in SO(X)$.

Sufficiency. Suppose $A \in SO(X)$. Then by Lemma 4, we have $\mathrm{Int}\, A \neq \emptyset$ because $A \neq \emptyset$. Hence it follows from $\mathrm{Int}\, A \subset \prod_{j=1}^{n} \mathrm{Int}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$ that $\prod_{j=1}^{n} \mathrm{Int}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha} X_\alpha \neq \emptyset$. Since $A \in SO(X)$, by Lemma 2 and 3, we obtain $\prod_{j=1}^{n} \mathrm{Cl}\,\mathrm{Int}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha = \mathrm{Cl}\,\mathrm{Int}\, A = \mathrm{Cl}\, A = \prod_{j=1}^{n} \mathrm{Cl}\, A_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} X_\alpha$. Therefore, we obtain $\mathrm{Cl}\,\mathrm{Int}\, A_{\alpha_j} = \mathrm{Cl}\, A_{\alpha_j}$ for each j $(1 \leqq j \leqq n)$. Again by Lemma 2, we obtain $A_{\alpha_j} \in SO(X_\alpha)$ for each j $(1 \leqq j \leqq n)$.

3. Semi-continuous mappings

THEOREM 3. – *If $f: X \to Y$ is a semi-continuous mapping and X_0 is an open set in X, then the restriction $f \mid X_0 : X_0 \to Y$ is semi-continuous.*

Proof. Since f is semi-continuous, for any open set V in Y, $f^{-1}(V)$ is semi-open in X. Hence by Lemma 1, $f^{-1}(V) \cap X_0$ is semi-open in X because X_0 is open. Therefore, by Theorem 6 of [3], $(f \mid X_0)^{-1}(V) = f^{-1}(V) \cap X_0$ is semi-open in X_0. This implies that $f \mid X_0$ is semi-continuous.

Remark. In Theorem 3, if $X_0 \in SO(X)$, then $f \mid X_0$ is not always semi-continuous, as shown by the following example due to N. Levine [3, Example 8].

Example. Let X and Y be the closed interval $[0, 1]$ with the usual topology and X_0 be $[1/2, 1]$. Let $f: X \to Y$ be a mapping as follows: $f(x) = 1$ if $0 \leqq x \leqq 1/2$ and $f(x) = 0$ if $1/2 < x \leqq 1$. Then f is semi-continuous. However, $(1/2, 1]$ is open in Y and $f^{-1}((1/2, 1]) \cap X_0 = \{1/2\} \notin SO(X_0)$. Therefore, $f \mid X_0$ is not semi-continuous.

THEOREM 4. – *Let $f: X \to Y$ be a mapping and $\{A_\alpha \mid \alpha \in \mathfrak{A}\}$ a semi-open cover of X, that is to say, $A_\alpha \in SO(X)$ for each $\alpha \in \mathfrak{A}$ and $\bigcup_{\alpha \in \mathfrak{A}} A_\alpha = X$. If the restriction $f \mid A_\alpha : A_\alpha \to Y$ is semi-continuous for each $\alpha \in \mathfrak{A}$, then f is semi-continuous.*

Proof. Suppose V is an arbitrary open set in Y. Then for each $\alpha \in \mathcal{A}$, we have $(f \mid A_\alpha)^{-1}(V) = f^{-1}(V) \cap A_\alpha \in SO(A_\alpha)$ because $f \mid A_\alpha$ is semi-continuous. Hence by Theorem 1, $f^{-1}(V) \cap A_\alpha \in SO(X)$ for each $\alpha \in \mathcal{A}$. By Theorem 2 of [3], we obtain that $\bigcup_{\alpha \in \mathcal{A}} f^{-1}(V) \cap A_\alpha = f^{-1}(V) \in SO(X)$. This implies that f is semi-continuous.

The following theorem is a generalization of Theorem B.

THEOREM 5. – *Let* $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ *and* $\{Y_\alpha \mid \alpha \in \mathcal{A}\}$ *be any two families of topological spaces with the same index set* $\mathcal{A}$. *For each* $\alpha \in \mathcal{A}$, *let* $f_\alpha : X_\alpha \to Y_\alpha$ *be a mapping. Then, a mapping* $f : \Pi X_\alpha \to \Pi Y_\alpha$ *defined by* $f((x_\alpha)) = (f_\alpha(x_\alpha))$ *is semi-continuous if and only if* f_α *is semi-continuous for each* $\alpha \in \mathcal{A}$.

Proof. Sufficiency. Suppose V is a basic open set of the topology of ΠY_α. Then there are $\alpha_j \in \mathcal{A}$ $(1 \leqq j \leqq n)$ and open sets V_{α_j} in Y_{α_j} such that $V = \prod_{j=1}^{n} V_{\alpha_j} \times \prod_{\alpha \neq \alpha_j} Y_\alpha$. Since f_{α_j} is semi-continuous, $f_{\alpha_j}^{-1}(V_{\alpha_j})$ is semi-open in X_{α_j} for each j $(1 \leqq j \leqq n)$. If there exists α_j such that $f_{\alpha_j}^{-1}(V_{\alpha_j}) = \varnothing$, then $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha = \varnothing$. Hence $f^{-1}(V)$ is semi-open in ΠX_α. If $f_{\alpha_j}^{-1}(V_{\alpha_j}) \neq \varnothing$ for all j $(1 \leqq j \leqq n)$, then $\prod_{j=1}^{n} f_{\alpha_j}^{-1}(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha \neq \varnothing$. Hence by Theorem 2, $f^{-1}(V) = \prod_{j=1}^{n} f_{\alpha_j}^{-1}(V_{\alpha_j}) \times \prod_{\alpha \neq \alpha_j} X_\alpha$ is semi-open in ΠX_α. Now for any open set W in Y, there exists a family $\{V_\lambda \mid \lambda \in \Delta\}$ of basic open sets such that $W = \bigcup_{\lambda \in \Delta} V_\lambda$. Hence by Theorem 2 of [3], $f^{-1}(W) = \bigcup_{\lambda \in \Delta} f^{-1}(V_\lambda)$ is semi-open in ΠX_α. This implies that f is semi-continuous.

Necessity. For each fixed $\alpha \in \mathcal{A}$, let $p_\alpha : \Pi Y_\beta \to Y_\alpha$ be the projection. Suppose V_α is an arbitrary open set in Y_α. Then $p_\alpha^{-1}(V_\alpha) = V_\alpha \times \prod_{\beta \neq \alpha} Y_\beta$ is open in ΠY_β. Since f is semi-continuous, $f^{-1}[p_\alpha^{-1}(V_\alpha)] = f_\alpha^{-1}(V_\alpha) \times \prod_{\beta \neq \alpha} X_\beta$ is semi-open in ΠX_β. If $f_\alpha^{-1}(V_\alpha)$ is empty, then it is obvious that f_α is semi-continuous. If $f_\alpha^{-1}(V_\alpha)$ is not empty, then $f_\alpha^{-1}(V_\alpha) \times \prod_{\beta \neq \alpha} X_\beta \neq \varnothing$ and hence by Theorem 2, $f_\alpha^{-1}(V_\alpha)$ is semi-open in X_α. This implies that f_α is semi-continuous.

The following theorem is a generalization of Theorem 15 of [3].

THEOREM 6. – *Let* $\{X_\alpha \mid \alpha \in \mathcal{A}\}$ *be any family of topological spaces. If* $f : X \to \Pi X_\alpha$ *is a semi-continuous mapping, then* $p_\alpha \circ f : X \to X_\alpha$ *is semi-continuous for each* $\alpha \in \mathcal{A}$, *where* p_α *is the projection of* ΠX_β *onto* X_α.

Proof. We shall consider a fixed $\alpha \in \mathcal{A}$. Suppose U_α is an arbitrary open set in X_α. Then $p_\alpha^{-1}(U_\alpha)$ is open in ΠX_α. Since f is semi-continuous, we have $f^{-1}[p_\alpha^{-1}(U_\alpha)] = (p_\alpha \circ f)^{-1}(U_\alpha) \in SO(X)$. Therefore, $p_\alpha \circ f$ is semi-continuous.

D. R. Anderson and J. A. Jensen [1] showed that if $f: X \to Y$ is a continuous and open mapping, then $f^{-1}(B) \in SO(X)$ for every $B \in SO(Y)$. The following theorem is a slight improvement of this theorem.

THEOREM 7. – *If $f: X \to Y$ is an open and semi-continuous mapping, then $f^{-1}(B) \in SO(X)$ for every $B \in SO(Y)$.*

Proof. For an arbitrary $B \in SO(Y)$, there exists an open set V in Y such that $V \subset B \subset \mathrm{Cl}\, V$. Since f is open, we have $f^{-1}(V) \subset f^{-1}(B) \subset$ $\subset f^{-1}(\mathrm{Cl}\, V) \subset \mathrm{Cl}\,[f^{-1}(V)]$ [4, (i), p. 13]. Since f is semi-continuous and V is open in Y, $f^{-1}(V) \in SO(X)$. Therefore by Theorem 3 of [3], we obtain $f^{-1}(B) \in SO(X)$.

By Remark 12 of [3], the composition mapping of two semi-continuous mappings is not always semi-continuous. However, we obtain the following corollary as an immediate consequence of Theorem 7.

COROLLARY. – *Let X, Y and Z be three topological spaces. If $f: X \to Y$ is an open and semi-continuous mapping and $g: Y \to Z$ is a semi-continuous mapping, then $g \circ f: X \to Z$ is semi-continuous.*

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