Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

S. S. Singh

Some theorems on Kählerian spaces with parallel Bochner curvature tensor

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **54** (1973), n.2, p. 205–209.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_2_205_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1973.

Geometria differenziale. — *Some theorems on Kählerian spaces with parallel Bochner curvature tensor.* Nota di S. S. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Proprietà degli spazi Kähleriani che ammettono un campo di tensori di curvatura paralleli secondo Bochner.

1. INTRODUCTION

An $n\,(=2\,m)$ dimensional Kählerian space K^n is a Riemannian space if it admits a structure tensor $\varphi_\mu{}^\lambda$ satisfying

$$\varphi_\mu{}^\alpha\,\varphi_\alpha{}^\lambda = -\,\delta_\mu{}^\lambda \tag{1.1}$$ (1)

where $$\delta_\mu{}^\lambda = \begin{cases} 1 & \lambda = \mu \\ 0 & \lambda \neq \mu, \end{cases}$$

$$\varphi_{\lambda\mu} = -\,\varphi_{\mu\lambda} \quad , \quad (\varphi_{\lambda\mu} = \varphi_\lambda{}^\alpha g_{\alpha\mu}) \tag{1.2}$$

and

$$\varphi_\lambda{}^k{}_{,\mu} = 0, \tag{1.3}$$

where the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

In a previous paper [3] (2) we have proved the following

THEOREM 1.1. *In a Kählerian projective symmetric space the scalar curvature is constant.*

THEOREM 1.2. *A Kählerian projective symmetric space is symmetric in the sense of Cartan.*

THEOREM 1.3. *A necessary and sufficient condition for a Kählerian space to be Kählerian projective symmetric is that the space be Kählerian symmetric in the sense of Cartan.*

(*) Nella seduta del 10 febbraio 1973.

(1) As to the notations we follow K. B. Lal and S. S. Singh [2] and S. S. Singh [3].

(2) The numbers in square brackets refer to the References given at the end of the paper.

2. Kählerian spaces with parallel Bochner curvature tensor

DEFINITION 2.1. *A Kählerian space for which the Bochner curvature tensor* $K_{\lambda\mu\nu}{}^{k}$ *defined by*

$$K_{\lambda\mu\nu}{}^{k} = R_{\lambda\mu\nu}{}^{k} + \frac{1}{n+4}\{R_{\lambda\nu}\delta_{\mu}^{k} - R_{\mu\nu}\delta_{\lambda}^{k} + g_{\lambda\nu}R_{\mu}^{k} - g_{\mu\nu}R_{\lambda}^{k} + S_{\lambda\nu}\varphi_{\mu}^{k} - S_{\mu\nu}\varphi_{\lambda}^{k} + \varphi_{\lambda\nu}S_{\mu}^{k} - \varphi_{\mu\nu}S_{\lambda}^{k} + 2S_{\lambda\mu}\varphi_{\nu}^{k} + 2S_{\nu}^{k}\varphi_{\lambda\mu}\} - \frac{R}{(n+2)(n+4)}\{g_{\lambda\nu}\delta_{\mu}^{k} - g_{\mu\nu}\delta_{\lambda}^{k} + \varphi_{\lambda\nu}\varphi_{\mu}^{k} - \varphi_{\mu\nu}\varphi_{\lambda}^{k} + 2\varphi_{\lambda\mu}\varphi_{\nu}^{k}\}, \tag{2.1}$$

satisfies

$$K_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = 0, \tag{2.2}$$

is called a Kählerian space with parallel Bochner curvature tensor.

We know that the Bochner curvature tensor of a space of constant holomorphic sectional curvature vanishes identically. Hence we have

Remark 2.1. A Kählerian space of constant holomorphic sectional curvature is a Kähler space with parallel Bochner curvature tensor.

Again, since $R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = 0$ implies $R_{\lambda\mu,\varepsilon} = 0$; which further implies $K_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = 0$; we have the following

Remark 2.2. A Kähler space which is symmetric in the sense of Cartan is a Kähler space with parallel Bochner curvature tensor.

DEFINITION 2.2. *A Kählerian space for which the holomorphically projective curvature tensor* $P_{\lambda\mu\nu}{}^{k}$ *defined by*

$$P_{\lambda\mu\nu}{}^{k} = R_{\lambda\mu\nu}{}^{k} + \frac{1}{n+2}\{R_{\lambda\nu}\delta_{\mu}^{k} - R_{\mu\nu}\delta_{\lambda}^{k} + S_{\lambda\nu}\varphi_{\mu}^{k} - S_{\mu\nu}\varphi_{\lambda}^{k} + 2S_{\lambda\mu}\varphi_{\nu}^{k}\}, \tag{2.3}$$

satisfies

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = 0, \tag{2.4}$$

has been called a Kählerian projective symmetric space (3).

If we put

$$L_{\lambda\mu} = R_{\lambda\mu} - \frac{R}{2(n+2)}g_{\lambda\mu} \tag{2.5}$$

and

$$M_{\lambda\mu} = \varphi_{\lambda}^{\alpha}L_{\alpha\mu} = S_{\lambda\mu} - \frac{R}{2(n+2)}\varphi_{\lambda\mu}, \tag{2.6}$$

(3) K. B. Lal and S. S. Singh [2].

$K_{\lambda\mu\nu}{}^{k}$ has the following form:

$$(2.7)\qquad K_{\lambda\mu\nu}{}^{k} = R_{\lambda\mu\nu}{}^{k} + \frac{1}{n+4}[L_{\lambda\nu}\,\delta_{\mu}^{k} - L_{\mu\nu}\,\delta_{\lambda}^{k} + g_{\lambda\nu}\,L_{\mu}{}^{k} - g_{\mu\nu}\,L_{\lambda}{}^{k} + M_{\lambda\nu}\,\varphi_{\mu}{}^{k} - M_{\mu\nu}\,\varphi_{\lambda}{}^{k} + \varphi_{\lambda\nu}\,M_{\mu}{}^{k} - \varphi_{\mu\nu}\,M_{\lambda}{}^{k} + 2\,M_{\lambda\mu}\,\varphi_{\nu}{}^{k} + 2\,\varphi_{\lambda\mu}\,M_{\nu}{}^{k}]\,.$$

THEOREM 2.1. *A Kählerian space* K^n *with parallel Bochner curvature tensor is symmetric in the sense of Cartan.*

Proof. Multiplying (2.7) by $g_{k\omega}$ and differentiating with respect to x^{ε} we get

$$(2.8)\qquad K_{\lambda\mu\nu\omega,\varepsilon} = R_{\lambda\mu\nu\omega,\varepsilon} + \frac{1}{n+4}[g_{\mu\omega}\,L_{\lambda\nu,\varepsilon} - g_{\lambda\omega}\,L_{\mu\nu,\varepsilon} + g_{\lambda\nu}\,L_{\mu\omega,\varepsilon} - g_{\mu\nu}\,L_{\lambda\omega,\varepsilon} + \varphi_{\mu\omega}\,M_{\lambda\mu,\varepsilon} - \varphi_{\lambda\omega}\,M_{\mu\nu,\varepsilon} + \varphi_{\lambda\nu}\,M_{\mu\omega,\varepsilon} - \varphi_{\mu\nu}\,M_{\lambda\omega,\varepsilon} + 2\,\varphi_{\nu\omega}\,M_{\lambda\mu,\varepsilon} + 2\,\varphi_{\lambda\mu}\,M_{\nu\omega,\varepsilon}]\,.$$

Again, multiplying (2.8) by $g^{\mu\nu}$ and taking into account (2.2) we obtain

$$(2.9)\qquad R_{\lambda\omega,\varepsilon} + \frac{1}{n+4}[L_{\lambda\omega,\varepsilon} - g_{\lambda\omega}\,g^{\mu\nu}\,L_{\mu\nu,\varepsilon} + L_{\lambda\omega,\varepsilon} - nL_{\lambda\omega,\varepsilon} - \varphi_{\mu\omega}\,g^{\mu\nu}\,M_{\lambda\nu,\varepsilon} - \varphi_{\lambda\nu}\,M_{\mu\omega,\varepsilon}\,g^{\mu\nu} + 2\,\varphi_{\nu\omega}\,g^{\mu\nu}\,M_{\lambda\mu,\varepsilon} + 2\,\varphi_{\lambda\mu}\,g^{\mu\nu}\,M_{\nu\omega,\varepsilon}] = 0\,.$$

It has been verified that the metric tensor g_{ij} and the Ricci tensor R_{ij} are hybrid in i and j and thereby these tensors satisfy

$$(2.10)\qquad g_{\lambda\mu} = g_{\varepsilon\eta}\,\varphi_{\lambda}{}^{\varepsilon}\,\varphi_{\mu}{}^{\eta}$$

and

$$(2.11)\qquad R_{\lambda\mu} = R_{\varepsilon\eta}\,\varphi_{\lambda}{}^{\varepsilon}\,\varphi_{\mu}{}^{\eta}\,.$$

By virtue of (2.10) and (2.11) we have

$$(2.12)\qquad \varphi_{\mu\omega}\,g^{\mu\nu}\,M_{\lambda\nu,\varepsilon} = \varphi_{\lambda\nu}\,g^{\mu\nu}\,M_{\mu\omega,\varepsilon} = \varphi_{\nu\omega}\,g^{\mu\nu}\,M_{\lambda\mu,\varepsilon} = \varphi_{\lambda\mu}\,g^{\mu\nu}\,M_{\nu\omega,\varepsilon} = -\left\{R_{\lambda\omega,\varepsilon} - \frac{R_{,\varepsilon}\,g_{\lambda\omega}}{2(n+2)}\right\}.$$

On using the above relations in (2.9) and making some simplifications we obtain

$$(2.13)\qquad R_{\lambda\omega,\varepsilon} - \frac{R_{,\varepsilon}\,g_{\lambda\omega}}{2(n+2)} = 0$$

i.e.

$$(2.14)\qquad L_{\lambda\omega,\varepsilon} = 0\,.$$

Using (2.14) in (2.8) and taking into account (2.2) and (2.6) it follows that

$$R_{\lambda\mu\nu\omega,\varepsilon} = 0 ,$$

which shows that the space is Kählerian symmetric in the sense of Cartan. This proves the statement.

Since, an Einstein Kähler space is a special one of the Riemannian spaces, in which the Bochner curvature tensor reduces to $U_{\lambda\mu\nu}{}^{k}$ given by

$$U_{\lambda\mu\nu}{}^{k} \overset{\text{def}}{=} R_{\lambda\mu\nu}{}^{k} + \frac{R}{n(n+2)} [g_{\lambda\nu}\,\delta_{\mu}^{k} - g_{\mu\nu}\,\delta_{\lambda}^{k} + \varphi_{\lambda\nu}\,\varphi_{\mu}{}^{k} - \varphi_{\mu\nu}\,\varphi_{\lambda}{}^{k} + 2\,\varphi_{\lambda\mu}\,\varphi_{\nu}{}^{k}] , \tag{2.15}$$

as immediate consequence we have the following

COROLLARY. *A Kähler Einstein space with parallel Bochner curvature tensor is symmetric in the sense of Cartan.*

By virtue of the above Theorem 2.1 and Remark 2.2 we can state

THEOREM 2.2. *A necessary and sufficient condition for a space* K^n *to be a Kähler space with parallel Bochner curvature tensor is that the space be Kählerian symmetric in the sense of Cartan.*

In view of Theorems 1.3 and 2.2 we have the following

THEOREM 2.3. *Kählerian spaces which are symmetric in the sense of Cartan are the only spaces which are both Kählerian projective symmetric and a Kähler space with parallel Bochner curvature tensor.*

Further, from Theorems 1.3 and 2.2 we also have

THEOREM 2.4. *A necessary and sufficient condition for a Kähler space* K^n *to be Kählerian projective symmetric is that* K^n *be a Kähler space with parallel Bochner curvature tensor.*

Moreover, from the Remark 2.1 and Theorem 2.1 we have

THEOREM 2.5. *A Kähler space of constant holomorphic sectional curvature is Kählerian symmetric in the sense of Cartan.*

Furthermore, the tensor $K_{\lambda\mu\nu}$ defined by

$$K_{\lambda\mu\nu} \overset{\text{def}}{=} R_{\mu\nu,\lambda} - R_{\lambda\nu,\mu} + \frac{1}{2(n+2)} (g_{\lambda\nu}\,\delta_{\mu}^{\varepsilon} - g_{\mu\nu}\,\delta_{\lambda}^{\varepsilon} + \varphi_{\lambda\nu}\,\varphi_{\mu}{}^{\varepsilon} - \varphi_{\mu\nu}\,\varphi_{\lambda}{}^{\varepsilon} + 2\,\varphi_{\lambda\mu}\,\varphi_{\nu}{}^{\varepsilon})\, R_{,\varepsilon} \tag{2.16}$$

satisfies the identity

$$K_{\lambda\mu\nu}{}^{\alpha}{}_{,\alpha} = \frac{n}{n+4}\, K_{\lambda\mu\nu} . \tag{2.17}$$

Therefore, on using (2.2) we have

THEOREM 2.6. *In a Kähler space with parallel Bochner curvature tensor the tensor* $K_{\lambda\mu\nu}$ *vanishes identically.*

3. Kählerian spaces with vanishing Bochner curvature tensor

DEFINITION 3.1. *A Kähler space for which the Bochner curvature tensor satisfies*

$$K_{\lambda\mu\nu}{}^{k} = 0 , \tag{3.1}$$

is called a Kähler space with vanishing Bochner curvature tensor.

Since, a Kählerian space with vanishing Bochner curvature tensor is a special one of the Kählerian spaces with parallel Bochner curvature tensor, as immediate consequences we can state the following theorems:

THEOREM 3.1. *A Kählerian space* K^n *with vanishing Bochner curvature tensor is Kählerian symmetric in the sense of Cartan.*

Proof. The proof follows from Theorem 2.1.

THEOREM 3.2. *A Kählerian space of constant holomorphic sectional curvature is a Kähler space with vanishing Bochner curvature tensor.*

Proof. The proof follows from the Remark 2.1.

THEOREM 3.3. *A Kähler space with vanishing Bochner curvature tensor is Kählerian projective symmetric.*

Proof. The statement follows from the Theorem 2.4.

Further, we have

Remark 3.1. The converse of the statement in Theorem 3.1 is not necessarily true.

Remark 3.2. The converse of the statement in Theorem 3.3 is not necessarily true.

The author is thankful to Prof. K. B. Lal for his helpful guidance during the preparation of this paper.

REFERENCES

[1] M. MATSUMOTO, *On Kählerian spaces with parallel or vanishing Bochner curvature tensor*, « Tensor », *20* (1969).
[2] K. B. LAL and S. S. SINGH, *On Kählerian spaces with recurrent Bochner curvature*, « Accademia Nazionale dei Lincei », *51* [3, 4] (1971).
[3] S. S. SINGH, *On Kählerian projective symmetric and Kählerian projective recurrent spaces* (Communicated for publication).
[4] S. TACHIBANA, *On the Bochner curvature tensor*, « Natural Science Report, Ochanomizu univ. » *18* (1), 15–19 (1967).
[5] K. YANO, *Differential geometry on complex and almost complex spaces*, « Pergamon Press (1965).