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David Lowell Lovelady

Convergent solutions of nonlinear differential equations

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Equazioni differenziali. — *Convergent solutions of nonlinear differential equations.* Nota di DAVID LOWELL LOVELADY, presentata [*] dal Socio G. SANSONE.

RIASSUNTO. — Sono date condizioni sufficienti che assicurano che se f è convergente allora ogni soluzione $v(t)$ dell'equazione

$$v'(t) = f(t) + F(t, v(t))$$

è convergente.

Questi risultati sono applicati per scegliere $\lim_{t\to\infty} v(t)$.

Soluzioni convergenti sono pure ottenute per l'equazione perturbata

$$u'(t) = f(t) + F(t, u(t)) + G(t, u(t)).$$

I. INTRODUCTION

Let Y be a Banach space with norm $|\ |$, and let R^+ be the set of all nonnegative real numbers. We shall obtain herein conditions which ensure that every solution v of

$$v'(t) = f(t) + F(t, v(t)) \tag{1}$$

is convergent (i.e., $\lim_{t\to\infty} v(t)$ exists), where F is a continuous function from $R^+ \times Y$ to Y and f is a convergent continuous function from R^+ to Y. The hypotheses which we shall place on F will be such that global existence and uniqueness for (1) follows from both of [13] and [17]. Our results on (1) will be related to the results of [12], and, indeed, the results of [12] will play a part in our proofs.

In § III we will indicate how the results of § II relate to terminal value problems for (1). In particular we will show that if z_0 is in Y then there is p_0 in Y such that $v(t) \to z_0$ if $f(t) \to p_0$.

We will also treat the perturbed equation

$$u'(t) = f(t) + F(t, u(t)) + G(t, u(t)) \tag{2}$$

and obtain conditions which ensure the convergence of its solutions. The technique in (2) will be to cite asymptotic equivalence results from [11] which ensure that (1) and (2) have the same asymptotic behavior. Thus asymptotic conclusions for (1) can be transferred to (2). Since the proofs in [11] required that Y be finitedimensional, we will impose that restriction when studying (2).

The problem of determining when solutions of ordinary differential equations are convergent has been studied by several Authors, usually for

(*) Nella seduta del 10 febbraio 1973.

linear equations with "small" nonlinear perturbations. See, for example, A. Wintner [19], [20], [21], F. Brauer [1], [2], T. G. Hallam [4], [5], Hallam, G. Ladas, and V. Lakshmikantham [7], Hallam and Lakshmikantham [8], Hallam and J. W. Heidel [6], and J. D. Mamedov [15]. Our techniques will be the use of differential inequalities of Lakshmikantham and S. Leela [9] and the notion of logarithmic derivative developed by T. Ważewski [18] and S. M. Lozinskii [14] and most recently employed by R. H. Martin, Jr. and the present Author [16], [13], [10], [11], [12].

II. Convergent solutions

Let φ be a continuous real-valued function on R^+. We shall need the following conditions for our theorems.

$(C1)$: If (t, x, y) is in $R^+ \times Y \times Y$ and c is a positive number then

$$|x - y - c\,[F(t, x) - F(t, y)]| \geq [1 - c\varphi(t)]\,|x - y|\,.$$

$(C2)$: There is a number M such that

$$\int_0^t \exp\left[\int_s^t \varphi(r)\,dr\right] ds \leq M$$

whenever t is in R^+.

$(C3)$: If x is in Y then $\lim_{t\to\infty} F(t, x)$ exists, and if K is a compact subset of Y then this convergence is uniform for x in K.

THEOREM 1. *Let f be a convergent continuous function from R^+ to Y, and suppose that each of $(C1)$, $(C2)$, and $(C3)$ is true. Then there is a member z of Y such that if v is a continuously differentiable function satisfying (1) on R^+ then*

$$\lim_{t\to\infty} v(t) = z\,.$$

It should be noted that in Theorem 1 the existence of $\lim_{t\to\infty} v(t)$ is part of the conclusion, not hypothesis. The theorem can be thought of as saying that every solution of (1) is convergent and all solutions converge to the same point. It is known that $(C1)$ implies that if x is in Y then there is exactly one continuously differentiable function v from R^+ to Y such that $v(0) = x$ and such that (1) is true whenever t is in R^+. This result was obtained independently by N. Pavel [17] and by R. H. Martin, Jr. and the present Author [13]. Before proving Theorem 1, we need the following lemma. Note that if $(C2)$ holds then M is necessarily positive.

LEMMA. *Suppose that $(C2)$ is true, and let $m = 1/M$. Then*

$$\liminf_{t\to\infty} \varphi(t) \leq -m\,.$$

Proof. It follows from [3, Lemma 1, p. 68] that there is a number k such that

$$\exp\left[\int_0^t \varphi(s)\, ds\right] \leq \exp(k - mt)$$

whenever t is in R^+. Thus

$$\int_0^t \varphi(s)\, ds \leq k - mt$$

whenever t is in R^+. Now, if b is in R^+ and ε is a positive number and $\varphi(t) \geq -m + \varepsilon$ whenever t is in $[b, \infty)$, then

$$\begin{aligned} k - mt &\geq \int_0^t \varphi(s)\, ds \\ &= \int_0^b \varphi(s)\, ds + \int_b^t \varphi(s)\, ds \\ &\geq \int_0^b \varphi(s)\, ds + (-m + \varepsilon)(t - b) \end{aligned}$$

whenever t is in $[b, \infty)$. This clearly cannot be, so the lemma is proved.

Proof of Theorem 1. Let A from Y to Y be given by $A(x) = \lim_{t \to \infty} F(t, x)$. Now, by (*C3*), A is continuous on compact subsets of Y and hence is continuous on Y. Also, if (x, y) is in $Y \times Y$ and c is a positive number,

$$\begin{aligned} |x - y - c[A(x) - A(y)]| &= \lim_{t \to \infty} |x - y - c[F(t, x) - F(t, y)]| \\ &\geq \limsup_{t \to \infty} [1 - c\varphi(t)]\, |x - y| \\ &\geq [1 + cm]\, |x - y| . \end{aligned}$$

It now follows from [16] that if p is in Y then there is exactly one member z of Y such that $p + A(z) = 0$. Let $p = \lim_{t \to \infty} f(t)$ and find z such that $p + A(z) = 0$. Thus $\lim_{t \to \infty} |f(t) + F(t, z)| = 0$. It follows from the results cited in the proof of the lemma that

$$\lim_{t \to \infty} \exp\left[\int_0^t \varphi(s)\, ds\right] = 0 , \tag{3}$$

and now an easy computation shows that

$$\lim_{t\to\infty} \int_0^t |f(s) + \mathrm{F}(x, z)| \exp\left[\int_s^t \varphi(r)\, \mathrm{d}r\right] \mathrm{d}s = 0 . \tag{4}$$

Now [12] says that (3) and (4) imply the conclusion of the theorem, and we are through.

III. Terminal value problems

It is clear in Theorem 1 that z depends on f, or, more specifically, on $\lim_{t\to\infty} f(t)$. This brings up the question of controlling (1) with chosen forcing functions, in order to reach a desired terminal value.

THEOREM 2. *Let each of (C1), (C2), and (C3) be true, and let z_0 be in* Y. *Then there is a member p_0 of* Y *such that, if f is a continuous function from* R^+ *to* Y *and $p_0 = \lim_{t\to\infty} f(t)$, then every solution v of* (1) *satisfies*

$$z_0 = \lim_{t\to\infty} v(t) .$$

Theorem 2 is clear from the construction in the proof of Theorem 1, and we shall not include a proof here.

IV. Stability of convergent solutions

From this point forward we shall assume that Y is finitedimensional. Let G be a continuous function from $\mathrm{R}^+ \times \mathrm{Y}$ to Y, and let ω be a continuous function from $\mathrm{R}^+ \times \mathrm{R}^+$ to R^+. We shall need the following conditions.

(*C4*): If (t, x) is in $\mathrm{R}^+ \times \mathrm{Y}$ then $|\mathrm{G}(t, x)| \le \omega(t, |x|)$.

(*C5*): If c is in R^+ then $\lim_{t\to\infty} \omega(t, c) = 0$.

(*C6*): If (r, s, t) is in $\mathrm{R}^+ \times \mathrm{R}^+ \times \mathrm{R}^+$ and $s \le t$ then $\omega(r, s) \le \omega(r, t)$.

THEOREM 3. *Suppose that each of (C1), (C2), (C4), (C5), and (C6) is true and that f is a continuous function from* R^+ *to* Y. *Then* (1) *and* (2) *are asymptotically equivalent in the sense that each of (i) and (ii) is true.*

(i): *If v is a bounded solution of* (1) *on* R^+ *then there is a member b of* R^+ *and a solution u of* (2) *on $[b, \infty)$ such that*

$$\lim_{t\to\infty} |v(t) - u(t)| = 0 . \tag{5}$$

(ii): *If b is in* R^+ *and u is a bounded solution of* (2) *on $[b, \infty)$ then there is a solution v of* (1) *on $[b, \infty)$ such that* (5) *is true.*

THEOREM 4. *Suppose that, in addition to the hypotheses of Theorem* 3, $(C3)$ *is true and* f *is convergent. Then there is a member* b *of* R^+ *such that there exists a convergent solution of* (2) *on* $[b, \infty)$. *Furthermore, if* b *is in* R^+ *and* u *is a bounded solution of* (2) *on* $[b, \infty)$ *then* u *is convergent.*

Theorem 3 follows easily from arguments so similar to the proof of [11, Theorem] that we shall not include them here. Theorem 4 follows directly from Theorems 1 and 3. Note that, in Theorem 4, all bounded (and hence convergent) solutions of (2) have the same limit, since (5) is true and we know from Theorem 1 that all solutions of (1) have the same limit.

The condition on the dimension of Y was imposed so as to permit the proof ideas of [11] to carry over to Theorem 3. The proof of [11, Theorem] involved applying the Schauder–Tychonoff theorem, in the form of [3, p. 9], to sets of Y-valued functions, and this requires Y to be finitedimensional.

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REFERENCES

[1] F. BRAUER, *Global behavior of solutions of ordinary differential equations*, « J. Math. Anal. Appl. », *2*, 145–158 (1961).
[2] F. BRAUER, *Bounds for solutions of ordinary differential equations*, « Proc. Amer. Math. Soc. », *14*, 36–43 (1963).
[3] W. A. COPPEL, *Stability and asymptotic behavior of differential equations*, D. C. Heath & Co., Boston 1965.
[4] T. G. HALLAM, *A terminal comparison principle for differential inequalities*, « Bull. Amer. Math. Soc. », *78*, 230–233 (1972).
[5] T. G. HALLAM, *Convergence of solutions of nonlinear differential equations*, « Ann. Mat. Pura Appl. », to appear.
[6] T. G. HALLAM and J. W. HEIDEL, *The asymptotic manifolds of a perturbed linear system of differential equations*, « Trans. Amer. Math. Soc. », *149*, 233–241 (1970).
[7] T. G. HALLAM, G. LADAS and V. LAKSHMIKANTHAM, *On the asymptotic behavior of functional differential equations*, « SIAM J. Math. Anal. », *3*, 58–64 (1972).
[8] T. G. HALLAM and V. LAKSHMIKANTHAM, *Growth estimates for convergent solutions of ordinary differential equations*, « J. Math. Phys. Sci. », *5*, 83–88 (1971).
[9] V. LAKSHMIKANTHAM and S. LEELA, *Differential and integral inequalities*, vol. 1, Academic Press, New York 1969.
[10] D. L. LOVELADY, *A functional differential equation in a Banach space*, « Funkcialaj Ekvacioj », *14*, 111–122 (1971).
[11] D. L. LOVELADY, *Asymptotic equivalence for two nonlinear systems*, « Math. Systems Theory », *7*, 170–175 (1973).
[12] D. L. LOVELADY, *Global attraction and asymptotic equilibrium for nonlinear ordinary equations*, Conference on the theory of ordinary and partial differential equations, pp. 303–307, « Lecture Notes Math. », *280*, Springer-Verlag, New York 1972.
[13] D. L. LOVELADY and R. H. MARTIN, JR., *A global existence theorem for an ordinary differential equation in a Banach space*, « Proc. Amer. Math. Soc. », *35*, 445–449 (1972).

[14] S. M. LOZINSKII, *Error estimates for the numerical integration of ordinary differential equations I*, « Isv. Vyss. Ucebn. Zaved. Mat. », (5) *6*, 52–90 (1958) (Russian).

[15] J. D. MAMEDOV, *One-sided estimates in the conditions for existence and uniqueness of solutions of the limit Cauchy problem in a Banach space*, « Sibirsk. Mat. Ž. », *6*, 1190–1196 (1965) (Russian).

[16] R. H. MARTIN, JR., *A global existence theorem for autonomous differential equations in a Banach space*, « Proc. Amer. Math. Soc. », *26*, 307–314 (1970).

[17] N. PAVEL, *Sur certaines équations differentielles abstraites*, « Boll. Un. Mat. Ital. », to appear.

[18] T. WAŻEWSKI, *Sur la limitation des integrales des systèmes d'équations différentielles linéaires ordinaires*, « Studia Math. », *10*, 48–59 (1948).

[19] A. WINTNER, *Asymptotic equilibria*, « Amer. J. Math. », *68*, 125–132 (1946).

[20] A. WINTNER, *An Abelian lemma concerning asymptotic equilibria*, « Amer. J. Math. », *68*, 451–454 (1946).

[21] A. WINTNER, *On a theorem of Bôcher in the theory of ordinary linear differential equations*, « Amer. J. Math. », *76*, 183–190 (1954).