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On Kählerian projective symmetric and Kählerian projective recurrent spaces

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Geometria differenziale. — *On Kählerian projective symmetric and Kählerian projective recurrent spaces.* Nota di S. S. SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — In una precedente Nota [1] ho definito e studiato spazi kähleriani simmetrici e spazi kähleriani ricorrenti. Questa Nota concerne lo studio di alcune proprietà di spazi kähleriani proiettivi simmetrici e kähleriani proiettivi ricorrenti.

1. INTRODUCTION

An $n\,(=2m)$ dimensional Kählerian space K^n is a Riemannian space which admits a tensor field $\varphi_\lambda{}^\mu$ satisfying

$$(1.1)^{(2)} \qquad \varphi_\alpha{}^\lambda \varphi_\mu{}^\alpha = -\delta_\mu^\lambda ,$$

$$(1.2) \qquad \varphi_{\lambda\mu} = -\varphi_{\mu\lambda} \quad , \quad (\varphi_{\lambda\mu} = g_{\mu\alpha}\varphi_\lambda{}^\alpha)$$

and

$$(1.3) \qquad \varphi_\lambda{}^\mu{}_{,k} = 0 ,$$

where the comma followed by an index denotes the operator of covariant differentiation with respect to the metric tensor g_{ij} of the Riemannian space.

The Riemannian curvature tensor is given by

$$R_{\lambda\mu\nu}{}^k = \partial_\lambda \left\{ {k \atop \mu\nu} \right\} - \partial_\mu \left\{ {k \atop \lambda\nu} \right\} + \left\{ {k \atop \lambda\alpha} \right\}\left\{ {\alpha \atop \mu\nu} \right\} - \left\{ {k \atop \mu\alpha} \right\}\left\{ {\alpha \atop \lambda\nu} \right\} {}^{(3)} ;$$

the Ricci tensor and the scalar curvature are respectively given by $R_{\lambda\mu} = R_{\alpha\lambda\mu}{}^\alpha$ and $R = g^{\lambda\mu} R_{\lambda\mu}$.

It is well known that these tensors satisfy the following identities [4]

$$(1.4) \qquad R_{\lambda\mu\nu}{}^\alpha{}_{,\alpha} = R_{\mu\nu,\lambda} - R_{\lambda\nu,\mu} ,$$

$$(1.5) \qquad R_{,\lambda} = 2\,R_\lambda{}^\alpha{}_{,\alpha} ,$$

$$(1.6) \qquad \varphi_\lambda{}^\alpha R_{\alpha\mu} = -R_{\lambda\alpha}\varphi_\mu{}^\alpha ,$$

$$(1.7) \qquad \varphi_\lambda{}^\alpha R_\alpha{}^k = R_\lambda{}^\alpha \varphi_\alpha{}^k .$$

(*) Nella seduta del 13 gennaio 1973.

(1) The numbers in square brackets refer to the references given at the end of the paper.

(2) All Latin and Greek indices run over the same range from 1 to n.

(3) $\partial_\lambda = \partial/\partial x^\lambda$, where $\{x^\lambda\}$ denotes real local coordinates.

We also have [2]

$$\varphi_\lambda^{\ \varepsilon} R_\varepsilon^{\ \mu} \varphi_\mu^{\ k} = - R_\lambda^{\ k} \tag{1.8}$$

and

$$\varphi_\lambda^{\ \alpha} R_\alpha^{\ \lambda} = 0 . \tag{1.9}$$

If we now define a tensor $S_{\mu\nu}$ by

$$S_{\mu\nu} = \varphi_\mu^{\ \alpha} R_{\alpha\nu} , \tag{1.10}$$

then we have

$$S_{\mu\nu} = - S_{\nu\mu} , \tag{1.11}$$

$$\varphi_\lambda^{\ \alpha} S_{\alpha\nu} = - S_{\lambda\alpha} \varphi_\nu^{\ \alpha} , \tag{1.12}$$

and

$$\varphi_\lambda^{\ \alpha} S_{\mu\nu,\alpha} = R_{\mu\lambda,\nu} - R_{\nu\lambda,\mu} . \tag{1.13}$$

The holomorphically projective curvature tensor is given by

$$P_{\lambda\mu\nu}^{\ \ \ \ k} = R_{\lambda\mu\nu}^{\ \ \ \ k} + \frac{1}{n+2} (R_{\lambda\nu} \delta_\mu^{\ k} - R_{\mu\nu} \delta_\lambda^{\ k} + S_{\lambda\nu} \varphi_\mu^{\ k} - S_{\mu\nu} \varphi_\lambda^{\ k} + 2 S_{\lambda\mu} \varphi_\nu^{\ k}) \tag{1.14}$$

whereas the Bochner curvature tensor $K_{\lambda\mu\nu\omega} = K_{\lambda\mu\nu}^{\ \ \ \ k} g_{\omega k}$ is defined by

$$\begin{aligned} K_{\lambda\mu\nu}^{\ \ \ \ k} = R_{\lambda\mu\nu}^{\ \ \ \ k} &+ \frac{1}{n+4} (R_{\lambda\nu} \delta_\mu^{\ k} - R_{\mu\nu} \delta_\lambda^{\ k} + g_{\lambda\nu} R_\mu^{\ k} - g_{\mu\nu} R_\lambda^{\ k} + \\ &+ S_{\lambda\nu} \varphi_\mu^{\ k} - S_{\mu\nu} \varphi_\lambda^{\ k} + \varphi_{\lambda\nu} S_\mu^{\ k} - \varphi_{\mu\nu} S_\lambda^{\ k} + 2 S_{\lambda\mu} \varphi_\nu^{\ k} + 2 \varphi_{\lambda\mu} S_\nu^{\ k}) - \\ &- \frac{R}{(n+2)(n+4)} (g_{\lambda\nu} \delta_\mu^{\ k} - g_{\mu\nu} \delta_\lambda^{\ k} + \varphi_{\lambda\nu} \varphi_\mu^{\ k} - \varphi_{\mu\nu} \varphi_\lambda^{\ k} + 2 \varphi_{\lambda\mu} \varphi_\nu^{\ k}) . \end{aligned} \tag{1.15}$$

2. Kählerian projective symmetric space

Definition 2.1. A Kählerian space satisfying

$$P_{\lambda\mu\nu\ \ ,\varepsilon}^{\ \ \ \ k} = 0 \qquad \text{or equivalently} \qquad P_{\lambda\mu\nu\omega,\varepsilon} = 0 \tag{2.1}$$

shall be called a Kählerian projective symmetric space.

Theorem 2.1. *In a Kählerian projective symmetric space the scalar curvature is a constant.*

Proof. From (1.14) and (2.1) we have

$$\begin{aligned} R_{\lambda\mu\nu\ \ ,\alpha}^{\ \ \ \ \alpha} = - \frac{1}{n+2} (R_{\lambda\nu,\alpha} \delta_\mu^{\ \alpha} - R_{\mu\nu,\alpha} \delta_\lambda^{\ \alpha} &+ \\ + S_{\lambda\nu,\alpha} \varphi_\mu^{\ \alpha} - S_{\mu\nu,\alpha} \varphi_\lambda^{\ \alpha} &+ 2 S_{\lambda\mu,\alpha} \varphi_\nu^{\ \alpha}) . \end{aligned} \tag{2.2}$$

Making some simplification with the help of (1.13) we get

$$(2.3) \qquad R_{\lambda\mu\nu}{}^{\alpha}{}_{,\alpha} = -\frac{4}{n+2}(R_{\lambda\nu,\mu} - R_{\mu\nu,\lambda}),$$

which on using (1.4) gives

$$(2.4) \qquad R_{\lambda\mu\nu}{}^{\alpha}{}_{,\alpha} = 0 \quad \text{when} \quad n \neq 2.$$

Multiplying (2.4) by $g^{\mu\nu}$ we obtain

$$R_{\lambda}{}^{\alpha}{}_{,\alpha} = 0.$$

Using this relation in (1.5) we get

$$R_{,\lambda} = 0$$

i.e. R is a constant.

THEOREM 2.2. *A Kählerian projective symmetric space is symmetric in the sense of Cartan.*

Proof. Multiplying (1.14) by $g_{k\omega}$ we get

$$(2.5) \quad P_{\lambda\mu\nu\omega} = R_{\lambda\mu\nu\omega} + \frac{1}{n+2}(R_{\lambda\nu}\, g_{\mu\omega} - R_{\mu\nu}\, g_{\lambda\omega} + S_{\lambda\nu}\, \varphi_{\mu\omega} - S_{\mu\nu}\, \varphi_{\lambda\omega} + 2\, S_{\lambda\mu}\, \varphi_{\nu\omega})$$

which in view of (2.1) gives

$$(2.6) \qquad R_{\lambda\mu\nu\omega,\varepsilon} = -\frac{1}{n+2}[R_{\lambda\nu,\varepsilon}\, g_{\mu\omega} - R_{\mu\nu,\varepsilon}\, g_{\lambda\omega} + $$
$$+ S_{\lambda\nu,\varepsilon}\, \varphi_{\mu\omega} - S_{\mu\nu,\varepsilon}\, \varphi_{\lambda\omega} + 2\, S_{\lambda\mu,\varepsilon}\, \varphi_{\nu\omega}].$$

Multiplying (2.6) by $g^{\mu\nu}$ and using theorem 2.1 we obtain

$$(2.7) \qquad \frac{n}{n+2} R_{\lambda\omega,\varepsilon} = 0$$

which implies that $R_{\lambda\omega,\varepsilon} = 0$. Hence from (2.6) it follows that $R_{\lambda\mu\nu\omega,\varepsilon} = 0$.

Therefore, the space is Kählerian symmetric in the sense of Cartan (4).

Remark. It may be remarked that a Kählerian symmetric space in the sense of Cartan is a Kählerian projective symmetric space.

In view of the above Remark and Theorem 2.2 we can state the following theorem:

THEOREM 2.3. *A necessary and sufficient condition for a Kählerian space to be Kählerian projective symmetric is that the space be Kählerian symmetric in the sense of Cartan.*

(4) K. B. LAL and S. S. SINGH [1].

3. KÄHLERIAN PROJECTIVE RECURRENT SPACE

DEFINITION 3.1. A Kähler space satisfying

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = a_\varepsilon P_{\lambda\mu\nu}{}^{k} \tag{3.1}$$

for some non-zero vector a_ε, shall be called a Kählerian projective recurrent space.

THEOREM 3.1. *A Kählerian recurrent space is Kählerian projective recurrent.*

Proof. A Kählerian recurrent space is characterized by

$$R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = a_\varepsilon R_{\lambda\mu\nu}{}^{k} \tag{3.2 a}$$

or equivalently by

$$R_{\lambda\mu\nu\omega,\varepsilon} = a_\varepsilon R_{\lambda\mu\nu\omega} \tag{3.2 b}$$

for some non-zero vector a_ε.

Multiplying (3.2 b) by $g^{\mu\nu}$ we get

$$R_{\lambda\omega,\varepsilon} = a_\varepsilon R_{\lambda\omega}.$$

From (1.14), in view of the above relation and (1.10) it follows that

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} = a_\varepsilon P_{\lambda\mu\nu}{}^{k}$$

which shows that the space is Kählerian projective recurrent.

THEOREM 3.2. *Every Kählerian projective recurrent space is a Kählerian recurrent space with Bochner curvature.*

Proof. Let the space be Kählerian projective recurrent.

Substituting in (3.1) from (1.14) we get

$$\begin{aligned} R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} + \frac{1}{n+2}(\delta_\mu^{\ k} R_{\lambda\nu,\varepsilon} - \delta_\lambda^{\ k} R_{\mu\nu,\varepsilon} + \varphi_\mu^{\ k} S_{\lambda\nu,\varepsilon} - \varphi_\lambda^{\ k} S_{\mu\nu,\varepsilon} + 2\varphi_\nu^{\ k} S_{\lambda\mu,\varepsilon}) = \\ = a_\varepsilon \left[R_{\lambda\mu\nu}{}^{k} + \frac{1}{n+2}(\delta_\mu^{\ k} R_{\lambda\nu} - \delta_\lambda^{\ k} R_{\mu\nu} + \varphi_\mu^{\ k} S_{\lambda\nu} - \varphi_\lambda^{\ k} S_{\mu\nu} + 2\varphi_\nu^{\ k} S_{\lambda\mu})\right]. \end{aligned} \tag{3.3}$$

Multiplying the above equation by $g^{\mu\nu}$ and making some simplifications with the help of (1.8), (1.9) and (1.10) we obtain

$$R_\lambda{}^{k}{}_{,\varepsilon} - a_\varepsilon R_\lambda{}^{k} = \frac{1}{n}(R_{,\varepsilon} - a_\varepsilon R)\,\delta_\lambda^{k}. \tag{3.4}$$

Multiplication by $g_{k\omega}$ reduces (3.4) to

$$R_{\lambda\omega,\varepsilon} - a_\varepsilon R_{\lambda\omega} = \frac{1}{n}(R_{,\varepsilon} - a_\varepsilon R)\,g_{\lambda\omega}. \tag{3.5}$$

Now from (1.15) we have

$$(3.6)\qquad K_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} = R_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} + \frac{1}{n+4}\,(\delta_\mu^{\ k} R_{\lambda\nu,\varepsilon} - \delta_\lambda^{\ k} R_{\mu\nu,\varepsilon} + g_{\lambda\nu} R_{\mu\ ,\varepsilon}^{\ k} -$$

$$- g_{\mu\nu} R_{\lambda\ ,\varepsilon}^{\ k} + \varphi_\mu^{\ k} S_{\lambda\nu,\varepsilon} - \varphi_\lambda^{\ k} S_{\mu\nu,\varepsilon} + \varphi_{\lambda\nu} S_{\mu\ ,\varepsilon}^{\ k} -$$

$$- \varphi_{\mu\nu} S_{\lambda\ ,\varepsilon}^{\ k} + 2\,\varphi_\nu^{\ k} S_{\lambda\mu,\varepsilon} + 2\,\varphi_{\lambda\mu} S_{\nu\ ,\varepsilon}^{\ k}) -$$

$$- \frac{R_{,\varepsilon}}{(n+2)(n+4)}\,(g_{\lambda\nu}\,\delta_\mu^{\ k} - g_{\mu\nu}\,\delta_\lambda^{\ k} + \varphi_{\lambda\nu}\,\varphi_\mu^{\ k} - \varphi_{\mu\nu}\,\varphi_\lambda^{\ k} + 2\,\varphi_{\lambda\mu}\,\varphi_\nu^{\ k}),$$

which on using (3.4) and (3.5) reduces to

$$(3.7)\qquad K_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon K_{\lambda\mu\nu}^{\ \ \ \ k} = R_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon R_{\lambda\mu\nu}^{\ \ \ \ k} + \frac{(R_{,\varepsilon} - a_\varepsilon R)}{n(n+2)}\,\cdot$$

$$\cdot [g_{\lambda\nu}\,\delta_\mu^{\ k} - g_{\mu\nu}\,\delta_\lambda^{\ k} + \varphi_{\lambda\nu}\,\varphi_\mu^{\ k} - \varphi_{\mu\nu}\,\varphi_\lambda^{\ k} + 2\,\varphi_{\lambda\mu}\,\varphi_\nu^{\ k}]\,.$$

Again the relation (3.1), after some simplification gives

$$(3.8)\qquad R_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon R_{\lambda\mu\nu}^{\ \ \ \ k} + \frac{(R_{,\varepsilon} - a_\varepsilon R)}{n(n+2)}\,\cdot$$

$$\cdot [g_{\lambda\nu}\,\delta_\mu^{\ k} - g_{\mu\nu}\,\delta_\lambda^{\ k} + \varphi_{\lambda\nu}\,\varphi_\mu^{\ k} - \varphi_{\mu\nu}\,\varphi_\lambda^{\ k} + 2\,\varphi_{\lambda\mu}\,\varphi_\nu^{\ k}] = 0\,.$$

From (3.7) in view of the above equation it follows that

$$K_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon K_{\lambda\mu\nu}^{\ \ \ \ k} = 0\,.$$

Hence the space is Kählerian recurrent with Bochner curvature.

THEOREM 3.3. *A necessary and sufficient condition for a Kählerian projective recurrent space to be Kählerian recurrent is that*

$$(3.9)\qquad R_{,\varepsilon} - a_\varepsilon R = 0$$

holds.

Proof. A Kählerian projective recurrent space satisfies the relation

$$(3.10)\qquad P_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon P_{\lambda\mu\nu}^{\ \ \ \ k} = 0\,.$$

Let the space be Kählerian recurrent, then we have

$$(3.11)\qquad R_{\lambda\mu\nu\ ,\varepsilon}^{\ \ \ \ k} - a_\varepsilon R_{\lambda\mu\nu}^{\ \ \ \ k} = 0$$

for some non-zero vector a_ε.

From (3.10) it follows that

$$R_{,\varepsilon} - a_\varepsilon R = 0\,.$$

Hence the condition is necessary.

Conversely, let (3.9) hold.

A Kählerian projective recurrent space satisfies the relation

$$R_{\lambda\mu,\varepsilon} - a_\varepsilon R_{\lambda\mu} = \frac{1}{n}(R_{,\varepsilon} - a_\varepsilon R)\, g_{\lambda\mu}. \tag{3.12}$$

which in view of (3.9) reduces to

$$R_{\lambda\mu,\varepsilon} - a_\varepsilon R_{\lambda\mu} = 0. \tag{3.13}$$

Using this relation in (3.1) it can be easily verified that

$$R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon R_{\lambda\mu\nu}{}^{k} = 0,$$

showing thereby that the space is Kählerian recurrent.

Therefore, the condition is sufficient. This completes the proof.

THEOREM 3.4. *A necessary and sufficient condition for a Kählerian projective recurrent space to be Kählerian recurrent is that the space be Kählerian Ricci–recurrent.*

Proof. Let us suppose that the Kählerian projective recurrent space is Kählerian recurrent, then evidently it is Kählerian Ricci–recurrent. Hence, the condition is necessary.

Conversely, let the space be Kählerian Ricci–recurrent. In such a space we have the relation

$$R_{\lambda\mu,\varepsilon} - a_\varepsilon R_{\lambda\mu} = 0. \tag{3.14}$$

Hence, from (1.14) in view of (3.14) it follows that

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon P_{\lambda\mu\nu}{}^{k} = R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon R_{\lambda\mu\nu}{}^{k}.$$

Using the fact that the space is Kählerian projective recurrent the above relation gives

$$R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon R_{\lambda\mu\nu}{}^{k} = 0 \tag{3.15}$$

which shows that the space is Kählerian recurrent.

Therefore, the condition is sufficient. This completes the proof.

THEOREM 3.5. *If a Kählerian space satisfies any two of the properties:*

(1) *The space is Kählerian projective recurrent;*

(2) *The space is Kählerian recurrent;*

(3) *The space is Kählerian Ricci–recurrent;*

it must also satisfy the third.

Proof. Kählerian projective recurrent, Kählerian recurrent and Kählerian Ricci-recurrent spaces are respectively characterized by the relations

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon P_{\lambda\mu\nu}{}^{k} = 0, \tag{3.16}$$

$$R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon R_{\lambda\mu\nu}{}^{k} = 0, \tag{3.17}$$

and

$$R_{\lambda\mu,\varepsilon} - a_\varepsilon R_{\lambda\mu} = 0. \tag{3.18}$$

Further, we have

$$P_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon P_{\lambda\mu\nu}{}^{k} = R_{\lambda\mu\nu}{}^{k}{}_{,\varepsilon} - a_\varepsilon R_{\lambda\mu\nu}{}^{k} + \tag{3.19}$$

$$+ \frac{1}{n+2} [\delta_\mu{}^k (R_{\lambda\nu,\varepsilon} - a_\varepsilon R_{\lambda\nu}) - \delta_\lambda{}^k (R_{\mu\nu,\varepsilon} - a_\varepsilon R_{\mu\nu}) +$$

$$+ \varphi_\mu{}^k (S_{\lambda\nu,\varepsilon} - a_\varepsilon S_{\lambda\nu}) - \varphi_\lambda{}^k (S_{\mu\nu,\varepsilon} - a_\varepsilon S_{\mu\nu}) + 2\varphi_\nu{}^k (S_{\lambda\mu,\varepsilon} - a_\varepsilon S_{\lambda\mu})].$$

The statement of the theorem follows in view of (3.16), (3.17), (3.18) and (3.19).

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