Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

# Rendiconti

S. P. Singh

## On a Theorem of Reinermann

*Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8*, Vol. **54** (1973), n.1, p. 46–48.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1973_8_54_1_46_0>

**Analisi funzionale.** — *On a Theorem of Reinermann* (*). Nota di S. P. SINGH, presentata (**) dal Corrisp. G. FICHERA.

RIASSUNTO. — Viene dato un teorema che generalizza quello di Reinermann sul punto unito di una trasformazione $g + h$ ($g$ non espansiva, $h$ fortemente continua). Da questa generalizzazione è possibile dedurre, come corollari, altri noti risultati.

Reinermann [7] has given the following theorem.

Let X be a uniformly convex Banach space and C be a non-empty, closed, bounded, convex subset of X. Let

$$f : C \to C,$$

$$g : C \to X,$$

$$h : C \to X,$$

be such that

*a)* $f = g + h$,

*b)* $\|g(x) - g(y)\| \leq \|x - y\|$ for all $x, y$ in C, ($g$ is called non-expansive),

*c)* $h$ is strongly continuous i.e. if $x_n$ converges to $x$ weakly then $hx_n$ converges to $hx$ strongly.

Then $f = g + h$ has at least one fixed point in C.

The aim of this note is to prove a more general theorem and to obtain a few known results as corollaries.

We need the following preliminaries:

Let X be a Banach space and let A be a bounded subset of X. We define $\alpha(A)$, the measure of non compactness of A, to be

*inf.* $\{\varepsilon > 0/A$ can be covered by a finite number of subsets of diameter $< \varepsilon\}$ (Kuratowski [5]).

Let $T : X \to X$ be a continuous mapping such that

$$\alpha(T(A)) \leq k \quad \alpha(A)$$

for any bounded subset A of X. Then T is called $k$–set contractive if $k < 1$. (Darbo, [3]). If $k = 1$, then T is called 1–set contraction. (Nussbaum [6]).

In case when $\alpha(T(A)) < \alpha(A)$, for any bounded subset A of X such that $\alpha(A) > 0$, then T is called densifying (Furi and Vignoli [4]).

(*) This research was partially supported by NRC–Grant A–3097.
(**) Nella seduta del 13 gennaio 1973.

A contraction mapping ($\|Tx - Ty\| \leq k \|x - y\|$ for all $x, y$ in X and $0 \leq k < 1$) and a completely continuous mapping (A continuous mapping that maps a bounded set into precompact set is called completely continuous.) are examples of densifying mappings. A non-expansive mapping is an example of 1-set contraction mapping.

Let C be a non-empty, closed, bounded and convex subset of a Banach space X and let $T : C \to C$ be a densifying mapping. Then T has at least one fixed point. (Furi and Vignoli [4]).

Let X be a reflexive Banach space and let C be a closed convex subset of X. Then every strongly continuous mapping $T : C \to X$, is completely continuous.

THEOREM 1. *Let* X *be a reflexive Banach space and let* C *be a non-empty, closed, bounded and convex subset of* X. *Let*

$$f : C \to C ,$$

$$g : C \to X ,$$

$$h : C \to X ,$$

*be such that*

*a)* $f = g + h$,

*b)* $g$ *is 1-set contraction and* $(1 - g)$ *is demi-closed* (1),

*c)* $h$ *is strongly continuous.*

*Then* $f$ *has at least one fixed point.*

*Remarks.* (i) If X is a Hilbert space and $g$ is non-expansive, then $1 - g$ is demi-closed [1].

(ii) If X is uniformly convex Banach space and $g$ is non-expansive, then $1 - g$ is demi-closed [2].

*Proof.* Let $k$ be a fixed positive number less than 1 and let $0 \in C$. Then the mapping $kg + kh$ is a densifying mapping and has a fixed point by a theorem of Furi and Vignoli [4]. Let $k_n$ be a sequence of numbers such that $0 \leq k_n < 1$, and $k_n \to 1$. Let $\{x_{k_n}\}$ be a sequence of points such that

$$k_n gx_{k_n} + k_n hx_{k_n} = x_{k_n} \quad ; \quad x_{k_n} \quad \text{in C.}$$

Since X is a reflexive Banach space and $\{x_{k_n}\}$ is bounded, therefore the sequence $\{x_{k_n}\}$ has a weakly convergent subsequence $\{x_{k_{n_i}}\}$ converging to $x$ in C. Then the fact that $x_{k_{n_i}} - gx_{k_{n_i}} - hx_{k_{n_i}} = (k_{n_i} - 1)(gx_{k_{n_i}} - hx_{k_{n_i}})$ and $hx_{k_{n_i}}$ converges strongly to $hx$ imply that $x_{k_{n_i}} - gx_{k_{n_i}}$ converges strongly to $hx$.

(1) If $x_i$ converges weakly to $x$ in C and $(1 - g) x_i$ converges strongly to $y$, then $(1 - g) x = y$.

By assumption $1 - g$ is demi-closed, therefore

$$(1 - g)\, x = hx,$$

i.e. $x = gx + hx = fx$.

Thus the proof.

COROLLARIES – 1. *In case* X *is a uniformly convex Banach space and* $g$ *is a non-expansive mapping, then we get the above quoted result due to Reinermann* [7].

A uniformly convex Banach space is reflexive and a non-expansive mapping is clearly 1-set contraction. $(1 - g)$ is demi-closed by Remark (ii).

– 2. *The following known result, due to Zabreiko, Kachurovsky and Krasnoselsky* [8], *can be obtained as a corollary to Theorem 1. Let* C *be a closed, bounded, convex subset of a Hilbert space* H. *Let* $f: C \to C$ *be a non-linear operator such that* $f = g + h$; *where* $g$ *is non-expansive mapping and* $h$ *is strongly continuous. Then* $f$ *has at least one fixed point in* C.

*Proof*. Clearly, a Hilbert space is uniformly convex and $g$ is non expansive. By Remark (ii) $(1 - g)$ is demi-closed and therefore all conditions of Theorem 1 are satisfied and $f$ has at least one fixed point in C.

## REFERENCES

[1] BROWDER F. E., *Non-expansive non-linear operators in a Banach space*, « Proc. Nat. Acad. Sci. U.S.A. », *54*, 1041–1044 (1965).

[2] BROWDER F. E., *Semicontractive and semiaccretive non-linear mappings in Banach spaces*, « Bull. Amer. Math. Soc. », *74*, 660–665 (1968).

[3] DARBO G., *Punti uniti in trasformazioni a codominio non compatto*, « Rend. del Sem. Mat. Univ. Padova », *24*, 84–92 (1955).

[4] FURI M. and VIGNOLI A., *On α-non-expansive mappings and Fixed points*, « Rend. Acc. Acc. Naz. Lincei », *18* (2), 195–198 (1970).

[5] KURATOWSKI K., *Topology*, *1* (New York, 1966).

[6] NUSSBAUM R. D., *The fixed point index and fixed point theorems for k-set contractions*, Ph. D. Thesis, Chicago (1969).

[7] REINERMANN, J., *Fixpunktsatze vom Krasnoselski-Typ*, « Math. Zeit. », *119*, 339–344 (1971).

[8] ZABREIKO P. P., KACHUROVSKY R. I. and KRASNOSELSKY M. A., *On a fixed point theorem for operators in Hilbert space*, « Funktion Analiz Priloz », *1*, 93–94 (1967).