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Periodic Solutions of a certain third order differential equation

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Equazioni differenziali ordinarie. — *Periodic Solutions of a certain third order differential equation.* Nota di JAMES OKOYE CHUKUKA EZEILO, presentata (*) dal Socio G. SANSONE.

RIASSUNTO. — L'Autore dimostra un teorema che assicura l'esistenza di almeno una soluzione periodica dell'equazione $\dddot{x} + g(\dot{x})\ddot{x} + b\dot{x} + h(x, \dot{x}, \ddot{x}) = p(t)$.

1. Consider the differential equation

$$\dddot{x} + g(\dot{x})\ddot{x} + b\dot{x} + f(x) = p(t) \tag{1.1}$$

where $b \geq 0$ is a constant and f, g, p are continuous functions. The special case of this equation in which f is bounded, that is $|f(x)| \leq F$ (constant) for all x, has been examined by a number of authors for the boundedness of solutions and, where p is periodic, for the existence of periodic solutions. Reissig [1], for example, generalizing an earlier result of mine [5], showed that if $f(x) \operatorname{sgn} x > 0$ $(|x| \geq x_0)$ and $G(y) \operatorname{sgn} y \to +\infty$ as $|y| \to \infty$ $\left(G(y) \equiv \int_0^y g(\eta)\, d\eta\right)$ and if $P(t) \equiv \int_0^t p(\tau)\, d\tau$ is bounded then solutions of (1.1) are all ultimately bounded, with the bound independent of solutions. On the periodic side there has not been as much success under the same general conditions on f and g, and the only results so far available, on the existence of periodic solutions, concern the two special cases: (i) $g \equiv$ constant ([2], [4], [6]), (ii) g, not necessarily constant, but such that

$$c|y| \geq G(y) \operatorname{sgn} y \geq a|y| \qquad (|y| \geq Y_0) \tag{1.2}$$

with a, c positive constants [3].

The object if the present Note is to free G of the restriction on the left in (1.2), and to extend the result, anyway, to the more general equation

$$\dddot{x} + g(\dot{x})\ddot{x} + b\dot{x} + h(x, \dot{x}, \ddot{x}) = p(t) \tag{1.3}$$

where h is a continuous function depending on all the arguments shown and, like the function f in (1.1), is bounded, that is

$$|h(x, y, z)| \leq H \qquad (\text{H constant}) \tag{1.4}$$

for all x, y and z.

(*) Nella seduta del 13 gennaio 1973.

Our result is summarized in the following theorem:

THEOREM. *Let h satisfy* (1.4) *and let the periodic function p with period ω be such that* $| P(t) | \leq A < \infty$ $(t \geq 0)$ *for some constant* A. *Suppose further that*

$$h(x, y, z) \operatorname{sgn} x > 0 \qquad (|x| \geq X_0) \tag{1.5}$$

$$G(y) \operatorname{sgn} y \geq a|y| \qquad (|y| \geq Y_0) \tag{1.6}$$

for some constants $a > 0$, X_0, Y_0.

Then the equation (1.3) *admits of at least one periodic solution with period* ω.

2. The proof is by the Leray–Schauder fixed point technique, a convenient starting point of which is the parameter (μ) – dependent equation

$$\dddot{x} + \{(1-\mu)a + \mu g(\dot{x})\}\ddot{x} + b\dot{x} + (1-\mu)cx + \mu h(x, \dot{x}, \ddot{x}) = \mu p. \tag{2.1}$$

which reduces to a linear equation when $\mu = 0$ and to the original equation (1.3) when $\mu = 1$. Here c is a positive constant to be fixed such that the linear equation corresponding to $\mu = 0$ is asymptotically stable.

To establish the theorem it will be enough to show that there is a choice of c such that all solutions of (2.1) are ultimately bounded, with the bound independent of solutions or of μ $(0 \leq \mu \leq 1)$. Unfortunately, however, the presence of the term

$$h_\mu \equiv (1-\mu)cx + \mu h(x, \dot{x}, \ddot{x})$$

which is bounded when $\mu = 1$ and unbounded otherwise makes the actual investigation of the boundedness of solutions of (2.1) quite an intractable one. A way out, suggested by Reissig's treatment in [3], is to consider, instead, the equation (2.1) with h_μ replaced by the function

$$h_\mu^* \equiv \mu h(x, \dot{x}, \ddot{x}) + (1-\mu)c\chi_R(x) \tag{2.2}$$

where

$$\chi_R = \begin{cases} x, & \text{if } |x| \leq R, \\ R \operatorname{sgn} x, & \text{if } |x| \geq R. \end{cases}$$

The function h_μ^* is bounded: in fact, by (1.4)

$$| h_\mu^* | \leq H + cR; \tag{2.3}$$

and therefore the new equation

$$\dddot{x} + \{(1-\mu)a + \mu g(\dot{x})\}\ddot{x} + b\dot{x} + h_\mu^* = \mu p \tag{2.4}$$

can be investigated readily for the boundedness of its solutions using for example, the techniques in [7]. The problem now, as far as the theorem is concerned, would be to prove not just that (I) the bound, so obtained, for solutions of (2.4) is independent of solutions and of μ $(0 \leq \mu \leq 1)$, but also that (II) for some suitably fixed R, every solution $x(t)$ of (2.4) ultimately satisfies $|x(t)| \leq R$. For, indeed, since $h_\mu^* = h_\mu$ when $|x| \leq R$, (II) means in effect that the actual Leray–Schauder fixed point considerations can be switched back from (2.4) to (2.1), so that the existence of a periodic solution of (2.1), and thus of (1.3), is implied in the usual way by (I).

3. SOME REMARKS ON NOTATION. Let $\bar{G}(\eta) \equiv \max_{|y|\leq\eta} |G(y)|$. We have, as a result of (1.6), that

$$yG(y) \geq ay^2 - d, \quad \text{for all } y, \tag{3.1}$$

where $d \equiv Y_0 \bar{G}(Y_0) + aY_0^2$.

In what follows $\delta, \delta_0, \delta_1, \cdots$ (without any arguments) stand for positive constants whose magnitudes depend only on a, b, d, A and H, but definitely not on μ or on R. Any δ, numbered or not, with some argument(s) displayed is a positive constant whose magnitude depends on a, b, d, A and H as well as the argument(s) explicitly displayed: thus, for example, $\delta(T_0, \rho_0)$ is a positive constant whose magnitude depends only on a, b, d, A, H, T_0 and ρ_0. The δ's are not the same in each place unless numbered, but all δ's (with or without arguments) having suffixes attached retain a fixed identity throughout.

4. BOUNDEDNESS OF SOLUTIONS OF (2.4). We shall in fact show that there are constants δ_0, δ_1 and a continuous function $\Delta(y)$ such that every solution $x(t)$ of (2.4) ultimately satisfies

$$|x(t)| \leq X_0 + 4\bar{G}(\Delta(\rho_0)) + \delta_1 \Delta(\rho_0) \tag{4.1}$$

$$|\dot{x}(t)| \leq \Delta(\rho_0) \quad , \quad |\ddot{x}(t)| \leq \delta_0 \Delta(\rho_0) + \bar{G}(\Delta(\rho_0)), \tag{4.2}$$

where X_0 is the constant in (1.5) and

$$\rho_0 \equiv H + cR + A + 1 . \tag{4.3}$$

We shall start with (4.2). For this it will be convenient to take (2.4) in the system form:

$$\dot{x} = y \quad , \quad \dot{y} = z - ay - \mu\{G(y) - ay\} + \mu P \quad , \quad \dot{z} = - h_\mu^* - by , \tag{4.4}$$

and sufficient to prove that every solution $(x(t), y(t), z(t))$ of (4.4) ultimately satisfies

$$y^2(t) + z^2(t) \leq \delta_2^2(\rho_0) \tag{4.5}$$

with $\delta_2(\rho_0)$ continuous in ρ_0.

The main tool for the proof of (4.5) is the continuous function

$$V = \frac{1}{2}(by^2 + z^2) - 2(\rho_0 + 1)W$$

where

$$W = \begin{cases} y \operatorname{sgn} z & \text{if } |z| \geq |y|, \\ z \operatorname{sgn} y & \text{if } |y| \geq |z|. \end{cases}$$

It is clear from the definition that

$$(4.6) \qquad -\delta_3 \rho_0^2 + \delta_4 (y^2 + z^2) \leq V \leq \delta_5 \rho_0^2 + \delta_6 (y^2 + z^2)$$

for some constants δ_3, δ_4, δ_5 and δ_6. Also if $(x(t), y(t), z(t))$ is any solution of (4.4) and

$$\dot{V}^* \equiv \limsup_{h \to +0} \frac{V(y(t+h), z(t+h)) - V(y(t), z(t))}{h},$$

we have, by an elementary differentiation that

$$(4.7) \qquad \dot{V}^* = -(1-\mu)aby^2 - \mu by G(y) + \mu by P - z h_\mu^* - $$
$$- 2(\rho_0 + 1)\{z - ay - \mu[G(y) - ay] + \mu P\} \operatorname{sgn} z$$

if $|z| \geq |y|$, and

$$(4.8) \qquad \dot{V}^* = -(1-\mu)aby^2 - \mu by G(y) + \mu by P - z h_\mu^* + $$
$$+ 2(\rho_0 + 1)(y + h_\mu^*) \operatorname{sgn} y$$

if $|y| \geq |z|$. Observe that the coefficient of μ, involving $G(y)$ on the right-hand side of (4.7) can be majorized by

$$-byG + \delta(\rho_0)|G|$$
$$= -\frac{3}{4} byG - \left\{\frac{1}{4} byG - \delta(\rho_0)|G|\right\}$$
$$\leq -\frac{3}{4} aby^2 + \delta(\rho_0),$$

by (3.1) and by virtue of the fact, resulting from (1.6), that the term in the curly bracket, which is continuous in y, is non negative if $|y| \geq \delta(\rho_0) \geq Y_0$. Thus (4.7) implies, in lieu of (2.3) and (4.3), that

$$(4.9) \qquad \dot{V}^* \leq -\frac{3}{4} aby^2 + \delta(\rho_0)(|y| + 1) - (\rho_0 + 2)|z|, \quad \text{if } |y| \leq |z|.$$

In the same way we have from (4.8), (3.1) and (2.3) that

$$(4.10) \qquad \dot{V}^* \leq -aby^2 + \delta(\rho_0)(|y| + 1) \quad \text{if } |y| \geq |z|.$$

The inequalities (4.9) and (4.10) show clearly that

$$\dot{V}^* \leq -1 \quad \text{if} \quad |y| \geq \delta_7(\rho_0) \tag{4.11}$$

for some sufficiently large $\delta_7(\rho_0)$. In the event, however, that $|y| < \delta_7(\rho_0)$, (4.9) shows too that, so long as $|z| \geq \delta_7(\rho_0)$,

$$\begin{aligned} \dot{V}^* &< \delta(\rho_0) - 2|z| \\ &< -1, \end{aligned}$$

provided further that $|z| \geq \delta(\rho_0)$ (sufficiently large). Thus

$$\dot{V}^* \leq -1, \quad \text{if} \quad |y| < \delta_7(\rho_0) \quad \text{provided that} \quad |z| \geq \delta_8(\rho_0),$$

which, when combined with (4.11), shows that

$$\dot{V}^* \leq -1 \quad \text{if} \quad y^2 + z^2 \geq \delta_9^2(\rho_0), \tag{4.12}$$

where $\delta_9^2(\rho_0) = \delta_7^2(\rho_0) + \delta_8^2(\rho_0)$.

The results (4.6) and (4.12) imply (4.5). To verify this note to begin with, that

$$y^2(t_0) + z^2(t_0) \leq \delta_9^2(\rho_0)$$

for some $t_0 \geq 0$, because otherwise we shall have that $\dot{V}^* \leq -1$ for all $t \geq 0$, which is incompatible with the lower bound restrict on V in (4.6). Next it is easy to see that $y^2(t) + z^2(t) \leq \delta_{10}^2(\rho_0)$ for all $t \geq t_0$, where

$$\delta_{10}^2(\rho_0) = \{(\delta_3 + \delta_5)\rho_0^2 + (\delta_4 + \delta_6)\delta_9^2(\rho_0)\}\delta_4^{-1}. \tag{4.13}$$

For, indeed, since $\delta_{10}(\rho_0) > \delta_9(\rho_0)$, the existence of a $T_0 > t_0$ such that

$$y^2(T_0) + z^2(T_0) > \delta_{10}^2(\rho_0)$$

would imply, $y^2(t) + z^2(t)$ being continuous, the existence of two instants t_1, t_2 $(t_2 > t_1 > t_0)$ such that

$$y^2(t_2) + z^2(t_2) = \delta_{10}^2(\rho_0) \quad , \quad y^2(t_1) + z^2(t_1) = \delta_9^2(\rho_0) \tag{4.14}$$

and

$$y^2(t) + z^2(t) \geq \delta_9^2(\rho_0) \qquad (t_1 \leq t \leq t_2). \tag{4.15}$$

However (4.15) and (4.12) imply that $V(t_2) < V(t_1)$ whereas (4.6) and (4.14) imply that

$$\begin{aligned} V(t_2) &\geq \delta_4\,\delta_{10}^2(\rho_0) - \delta_3\,\rho_0^2 \\ &> \delta_6\,\delta_9^2(\rho_0) + \delta_5\,\rho_0^2 \qquad \text{(by (4.13))} \\ &\geq V(t_1) \end{aligned}$$

which is a contradiction. Hence

$$y^2(t) + z^2(t) \leq \delta_{10}^2(\rho_0) \qquad (t \geq t_0).$$

It remains now only to add that each $\delta(\rho_0)$, with or without suffix, which has featured in the foregoing is a continuous function of ρ_0 to complete the verification of (4.5), and thus of (4.2).

We turn now to (4.1). The procedure here will be almost as in [7]. Let $x = x(t)$ be any solution of (2.4) and define $\psi = \psi(t)$ by

$$\psi = \ddot{x} + (1-\mu)a\dot{x} + \mu G(\dot{x}) + bx - \mu P(t). \tag{4.16}$$

Verify that $\dot{\psi} = -h_\mu^*$, so that by (1.5) and (2.2), $\dot{\psi}$ is strictly positive (or negative) when $x \leq -X_0$ (or $\geq X_0$). Also, by (4.2),

$$|\psi(t) - bx| \leq 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0), \qquad t \geq T_0 \tag{4.17}$$

for some $\delta_{11}(\rho_0)$ and T_0. We will now show, as a result, that

$$|\psi(t)| \leq bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0), \qquad t \geq T_1 \tag{4.18}$$

(for some $T_1 \geqq T_0$) from which (4.1) will then follow in lieu of our definition (4.16) of ψ.

We start by proving the existence of $T_2 \geq T_0$ such that

$$\psi(T_2) < bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0). \tag{4.19}$$

Suppose indeed that this were not the case so that

$$\psi(t) \geq bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0), \quad \text{for all } t \geq T_0. \tag{4.20}$$

Then, by (4.17), $x(t) \geq X_0$ and therefore, by the property of $\dot{\psi}$ stated earlier, $\psi(t) \leq \psi(T_0)$ for all $t \geq T_0$. Thus ψ is bounded and so we have not just $x(t) \geq X_0$, but indeed $X_0 \leq x(t) \leq \delta(T_0, \rho_0)$, for all $t \geq T_0$. However, since h_μ^* is continuous in the variables $x, \dot{x}, \ddot{x}$, the fact that each of $x, \dot{x}, \ddot{x}$ is bounded for all $t \geq T_0$ leads to the stronger estimate

$$\dot{\psi} = -h_\mu^* \leq -\delta(T_0, \rho_0) < 0 \qquad (t \geq T_0)$$

which is impossible in view of the lower bound restriction (4.20) on ψ. Thus there is a $T_2 \geq T_0$ for which (4.19) holds. We assert now that, corresponding to any such T_2,

$$\psi(t) \leq bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0), \qquad t \geq T_2. \tag{4.21}$$

For the existence of a $T_3 > T_2$ such that

$$\psi(T_3) > bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0)$$

implies that

$$\psi(T_4) = bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0) \quad \text{and} \quad \dot{\psi}(T_4) > 0 \tag{4.22}$$

for some T_4 in the open interval (T_2, T_3). However, if $\psi(T_4) = bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0)$ then, by (4.17), $x(T_4) \geq X_0$ so that as before

$\dot{\psi}(T_4) < 0$ which contradicts the second condition in (4.22). This proves (4.21).

By an entirely analogous reasoning it can also be shown that

$$\psi(t) \geq -\{bX_0 + 2\bar{G}(\Delta(\rho_0)) + \delta_{11}(\rho_0)\}$$

for all sufficiently large t. Thus (4.18) holds, and (4.1) then follows.

5. COMPLETION OF THE PROOF OF THE THEOREM. By the remarks at the end of § 2 it remains now only to show that there are constants $c > 0$ and $R > 0$ such that (i) *the linear system*

$$\dddot{x} + a\ddot{x} + b\dot{x} + cx = 0$$

is asymptotically stable, and such that also (ii) *every solution* $x(t)$ *of* (2.4) *ultimately satisfies* $|x(t)| \leq R$.

We observe that, by the Routh–Hurwitz stability criteria, only the condition $0 < c < ab$ is required to secure (i). Also, by (4.1) and (4.3), the condition (ii) would be met by any pair c and R which satisfy the inequality

$$R \geq \chi(cR + \delta_{12}) \tag{5.1}$$

where $\chi(y) \equiv X_0 + 4\bar{G}(\Delta(y)) + \delta_1\Delta(y)$ and $\delta_{12} = H + A + 1$. Now let

$$\bar{\chi}(r) \equiv \max_{|y| \leq r} \chi(y).$$

It is easy to check that (5.1) is satisfied, for example, by

$$R = \bar{\chi}(\delta_{12} + 1) \equiv \delta_{13} \quad , \quad c = (\delta_{13} + 2)^{-n} \tag{5.2}$$

for any real number $n \geq 1$. If we now fix n large enough to ensure that

$$(\delta_{13} + 2)^{-n} < ab$$

then the value of c given in (5.2) will also suffice for (i).

It is thus possible to fix c and R so that (i) and (ii) together are satisfied, and the theorem now follows, as indicated earlier.

6. A FURTHER GENERALIZATION. The present theorem is extendable to the case, investigated in an earlier paper [7] for boundedness only, in which the coefficient b in (1.3) is replaced by a continuous function $\varphi(x)$ satisfying

$$\Phi(x) - \gamma x = o(1) \quad \text{as} \quad |x| \to \infty \qquad \left(\Phi(x) \equiv \int_0^x \Phi(y)\,dy\right) \tag{6.1}$$

for some constant $\gamma > 0$, with g, h, p as before.

The procedure for the new equation is exactly as for the present (1.3) except that (2.1), (4.4) and (4.16) respectively would have to be replaced by

$$(2.1)^* \qquad \dddot{x} + \{(1-\mu)\,a + \mu g(\dot{x})\}\,\ddot{x} + \{(1-\mu)\,\gamma + \mu\varphi(x)\} + \\ + (1-\mu)\,cx + \mu h = \mu p$$

$$(4.4)^* \qquad \dot{x} = y \quad , \quad \dot{y} = x - ay - \mu\,\{G(y) - ay\} - \mu\,\{\Phi(x) - \gamma x\} + \mu P \\ \dot{z} = -h_\mu^* - \gamma y\,,$$

$$(4.16)^* \qquad \psi = \ddot{x} + \{(1-\mu)\,a\dot{x} + \mu G(\dot{x})\} + (1-\mu)\,\gamma x + \mu\Phi(x) - \mu P,$$

and the estimate (4.17) for $|\psi - bx|$ by an estimate for $|\psi - \gamma x|$ which should take into account the fact that $\Phi(x) - \gamma x$ *is bounded*, by (6.1).

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