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On the refinements of a banachical principal fibre bundle

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RENDICONTI

DELLE SEDUTE

DELLA ACCADEMIA NAZIONALE DEI LINCEI

Classe di Scienze fisiche, matematiche e naturali

Seduta del 13 gennaio 1973

Presiede il Presidente BENIAMINO SEGRE

SEZIONE I

(Matematica, meccanica, astronomia, geodesia e geofisica)

Matematica. — *On the refinements of a banachical principal fibre bundle.* Nota di LILIANA MAXIM, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Alcuni risultati recenti [5] sui raffinamenti d'uno spazio fibrato principale differenziabile vengono qui estesi dal caso finito al caso infinito (spazi di Banach reali).

In [5], D. Papuc defines the refinement notion for a differentiable principal fibre bundle (finite-dimensional) and investigates its differential and topological properties. Further on, we extend this configuration to the banachical principal fibre bundles.

All the manifolds met with in this work are of the C^∞-classes, modeled on Banach real spaces.

Let G be a group manifold ([2], 5.1.2) which acts on the right on a manifold P satisfying the following conditions:

a) G acts properly and freely on P,

b) $\forall x \in P$, the mapping $f_x : G \to P$ which maps g into gx, is an immersion.

Remark. If P is finite dimensional, the above condition *b*) is a consequence of the condition *a*) (see [2]).

The equivalence relation induced by G on P is, under these conditions, regular (see [2]); noting by B the quotient space of P, P/G, and by π the canonical projection at P on B, the quadruplet $\xi = (P, G, B, \pi)$ is a principal fibre bundle over B with group G.

(*) Nella seduta del 13 gennaio 1973.

We consider now a sequence of the closed group submanifolds in G ([2], 5.8.3); let this sequence be noted as in [5]:

$$\mathfrak{N} = (\mathrm{G} = \mathrm{H}_0 \supset \mathrm{H}_1 \supset \cdots \supset \mathrm{H}_n = \{e\})$$

where n is a natural number, $n \geq 2$, and e is the neutral element in G. We shall investigate the structure determined by the pair $(\xi , \mathfrak{N})$.

First of all, we remark that from [3] (ex. 1 § 4) and from [2] § 5.8.3 there results that any closed group submanifold in G satisfies the conditions *a*) and *b*); then $\mathrm{P}/\mathrm{H}_j = \mathrm{P}_j$ shall be the manifolds $\forall j$, and, noting by π_j the canonical projection, the quadruplets $\xi_j = (\mathrm{P} , \mathrm{H}_j , \mathrm{P}_j , \pi_j)$ are the principal fibre bundles $\forall j \, (0 \leq j \leq n)$. It is obvious that $\xi_0 = \xi$.

Also, if P is a group manifold, and G is a closed group submanifold in P which acts on P by the translations to the right, the conditions *a*) and *b*) are satisfied; the base space of the principal fibre bundle obtained in this way is a homogeneous space. Consequently, the quadruplets $\eta_j = (\mathrm{G} , \mathrm{H}_j , \mathrm{G}/\mathrm{H}_j , p_j) , \gamma_{jk} = (\mathrm{H}_j , \mathrm{H}_k , \mathrm{H}_j/\mathrm{H}_k , q_{jk})$ are the principal fibre bundles $\forall j , k \, (0 \leq j < k \leq n)$.

The following theorem proved in [4] leads to the same result:

THEOREM 1. $\forall j , k \, (0 \leq j < k \leq n)$, *the projection* $p_{jk} : \mathrm{G}/\mathrm{H}_k \rightarrow \mathrm{G}/\mathrm{H}_j$ *is a fibre bundle with the manifold* $\mathrm{H}_j/\mathrm{H}_k$ *as type fibre and* $\mathrm{G}_{jk} = \mathrm{H}_j/\mathrm{N}_{jk}$ *as structure group, where* N_{jk} *is the largest invariant subgroup in* H_j, *contained in* H_k.

Let $\eta_{jk} = (\mathrm{G}/\mathrm{H}_k , \mathrm{G}_{jk} , \mathrm{G}/\mathrm{H}_j , \mathrm{H}_j/\mathrm{H}_k , p_{jk})$ be these fibre bundles;

$$\eta_{j0} = \eta_j , \, p_{hj} \circ p_{jk} = p_{hk} \qquad \text{for} \quad 0 \leq h < j < k \leq n \qquad \text{and} \quad p_{jn} = p_j \, .$$

The following result is an extension of the Theorem 1 out of [5] to the banachic case.

THEOREM 2. $\forall j , k \, (0 \leq j < k \leq n)$, *the projection* $\pi_{jk} : \mathrm{P}_k \rightarrow \mathrm{P}_j$ *is a fibre bundle with* $\mathrm{H}_j/\mathrm{H}_k$ *as type fibre and* G_{jk} *as the structure group*. We denote this fibre bundle by $\xi_{jk} = (\mathrm{P}_k , \mathrm{G}_{jk} , \mathrm{P}_j , \mathrm{H}_j/\mathrm{H}_k , \pi_{jk})$.

Proof. Since $\pi_{hk} = \pi_{hj} \circ \pi_{jk} , \, \pi_{jn} = \pi_j$ and $\pi_{0n} = \pi , \forall h , j , k , (0 \leq h < < j < k \leq n)$, there results the commutativity of the following diagram:

$$\begin{tikzcd}
\mathrm{P} \arrow[rr, "\pi_k"] \arrow[dr, "\pi_j"] \arrow[dd, "\pi"'] & & \mathrm{P}_k \arrow[dl, "\pi_{jk}"] \\
& \mathrm{P}_j \arrow[dl, "\pi_{0j}"] & \\
\mathrm{B} = \mathrm{P}_0 & &
\end{tikzcd}$$

Moreover, π_j , π_k, being the surjective morphisms $\forall j , k , \pi_{jk}$ is a morphism at P_k on P_j.

We consider now a trivialisation atlas for each fibre bundle introduced above. Such that:

$$
\begin{array}{lll}
\mathcal{A}_j = (U^j, \Phi^i_U) & \text{for} & \xi_j\,, \\
\mathcal{A} = (U\,, \Phi_U) & \text{for} & \xi_0 = \xi\,, \\
a_j = (u^j\,, \varphi^j_u) & \text{for} & \eta_j\,, \\
a_{jk} = (u^{ik}, \varphi^{jk}_u) & \text{for} & \eta_{jk}\,.
\end{array}
$$

To each trivialisation open set U for ξ, we associate the following diagram:

$$
\begin{array}{rcccc}
 & U\times G & \xrightarrow{\ \Phi_U\ } & \pi^{-1}_{0n}(U) \subset P & \\
1_U \times p_k & \downarrow & & \downarrow & \pi_k \\
 & U\times G/H_k & \xrightarrow{\ \Phi_{U,k}\ } & \pi^{-1}_{0k}(U) \subset P_k & \\
1_U \times p_{jk} & \downarrow & & \downarrow & \pi_{jk} \\
 & U\times G/H_j & \xrightarrow{\ \Phi_{U,j}\ } & \pi^{-1}_{0j}(U) \subset P_j & \\
1_U \times p_{0j} & \swarrow & & \downarrow & \pi_{0j} \\
 & U & \xrightarrow{\ 1_U\ } & U \subset B &
\end{array}
$$

where $\Phi_{U,j}\,, \Phi_{U,k}$ are the maps uniquely determined by the condition that the diagram should be commutative. They are obviously diffeomorphisms. Moreover, for another trivialisation map $(U', \Phi_{U'})$ of the ξ, with $U \cap U' \neq \varnothing$ we have:

$$\Phi_{U',j} \circ \Phi_{U,j} = \alpha_j \in G/H_j \qquad \text{where} \quad p_j(\alpha_n) = \alpha_j\,.$$

Using the fibre structure of the η_{jk}, we get a trivialisation atlas for ξ_{jk} (as in [5]):

$$\mathcal{A}_{jk} = (U^{jk}, \Phi^{jk}_U)$$

where

$$U^{jk} = \Phi_{U,j}(U \times u^{jk})$$

$$\Phi^{jk}_U : U^{jk} \times H_j/H_k \to \pi^{-1}_{jk}(U^{jk})$$

$$\Phi^{jk}_U = \Phi_{U,k}/U \times \varphi^{jk}_U(u^{jk} \times H_j/H_k) \circ (1_U \times \varphi^{jk}_U) \circ \{[\Phi_{U,j}/U \times u^{jk}]^{-1} \times 1_{H_j/H_k}\}\,.$$

Since the proof of the fact that $\mathcal{A}_{jk}$ is a trivialisation atlas for a fibre structure is similar to that from [5], we will not introduce it here.

Remarks. 1) If $\mathfrak{N}$ is normal, the fibre bundles $\xi_{jj+1}\,, 0 \leq j \leq n$, are the principal fibre bundles with H_j/H_{j+1} as the structure group.

2) The previous results do not hold good if H_j are the group quasi-submanifolds in G (this notion is defined in [2] 5.8.3);

3) If P is a Hilbert manifold with a bundle type metric, the Theorem 2 is a consequence of some results from [4], devoted to the leaves on the banachic manifolds.

DEFINITION 1. The set of the fiber bundles $\xi_{jk}\,, 0 \leq j < k \leq n$, is called *the tissue* associate to the structure $(\xi\,, \mathfrak{N})$. The tern $(\xi\,, \xi_{0j}\,, \xi_j)$, $0 \leq j \leq n$, is called *the cell of the refinement* of ξ through the subgroup H_j.

It is obvious that the topological properties related to the homotopy chain of the bundles ξ_{0n}, ξ_{0j}, ξ_{jn}, associated to the refinement remarked in [5], remain available in this case.

We attempt a generalization of the geometrical properties related to the infinitesimal connections of the principal fibre bundles $\xi_{0n} = \xi$ and $\xi_{jn} = \xi_j$.

Therefore, we consider a refinement of the ξ through the subgroup H_j, (ξ, ξ_{0j}, ξ_j). Let $\gamma = \{\Gamma\}$ and $\gamma_j = \{\Gamma_j\}$ be the sets of the C^∞-connections on the principal fibre bundles ξ and ξ_j, respectively (this notion is defined in [6]).

A C^∞-connection Γ on the principal fibre bundle ξ is a C^∞-splitting of the following exact chain of the vector bundles over P:

$$0 \longrightarrow P \times \mathbf{G} \xrightarrow{\ I\ } TP \xrightarrow{T\pi!} \pi^* TB \longrightarrow 0$$

where: – $\mathbf{G}$ is the Lie algebra of the group G, $(P \times \mathbf{G}, G, p_1)$ is a G-fibre bundle on which G acts as follows:

$$(p, a) \cdot g = (p \cdot g, ad(g^{-1})(a)) \qquad \forall (p, a, g) \in P \times \mathbf{G} \times G$$

– π^*TB is the pull-back over π of the tangent vector bundle $(TB, B, \mathcal{T}_B)$; noting by E the model space for the manifold B, π^*TB is $Gl(E)$-fibre bundle.

– $T\pi!$ is defined by: $T\pi!(\omega) = (\mathcal{T}_P(\omega), T\pi(\omega))\ \forall \omega \in TP$, where $T\pi$ is the morphism of the tangent vector bundles $(TP, P, \mathcal{T}_P)$ and $(TB, B, \mathcal{T}_B)$.

The connexion Γ determines the following decomposition in the direct sum:

$$TP = I(P \times \mathbf{G}) \oplus \Gamma(\pi^* TB). \tag{1}$$

This decomposition defines a reduction of the structure group $Gl(E \times \mathbf{G})$ of the vector bundle $(TP, P, \mathcal{T}_P)$ to the subgroup $\begin{pmatrix} Gl(E) & 0 \\ 0 & G \end{pmatrix}$. We identify the group manifolds $Gl(E)$ and G with the closed group submanifolds $Gl(E) \times \{e\}$ and $\{1_E\} \times G$, respectively.

Similarly, a C^∞-connection Γ_j on the principal fibre bundle ξ_j is a C^∞-splitting of the following exact chain of the vector bundles over P:

$$0 \longrightarrow P \times \mathbf{H}_j \xrightarrow{\ I_j\ } TP \xrightarrow{T\pi_j!} \pi_j^* TP_j \longrightarrow 0.$$

Noting by E_j the model space of the manifold P_j, $\pi_j^* TP_j$ is a $Gl(E_j)$-fibre bundle; $\mathbf{H}_j$ is the Lie algebra of the group submanifold H_j and $(P \times \mathbf{H}_j, H_j, p_1)$ is a H_j-fibre bundle.

The connection Γ_j on ξ_j determines the decomposition:

$$TP = I_j(P \times \mathbf{H}_j) \oplus \Gamma_j(\pi_j^* TP_j)$$

which defines a reduction of the structure group of the vector bundle $(TP, P, \mathcal{T}_P)$ to the subgroup $\begin{pmatrix} Gl(E_j) & 0 \\ 0 & H_j \end{pmatrix}$. We identify the group manifolds $Gl(E_j)$ and H_j with the closed group submanifolds $Gl(E_j) \times \{e\}$ and $\{1_{E_j}\} \times H_j$.

We get the following result:

THEOREM 3. *If the base space* B *of the principal fibre bundle* ξ *is a parallelisable manifold, a fixed connection* Γ_j *on* ξ_j *defines a mapping* $h_{\Gamma_j}: \gamma \to \gamma_j$.

Proof. Let Γ be a C^∞-connection on ξ; it defines the decomposition (1) and a reduction of the structure group of TP to the subgroup $\begin{pmatrix} Gl(E) & O \\ O & G \end{pmatrix}$ which, in our hypothesis, is $\{1_E\} \times G$. The fixed connection Γ_j on ξ_j defines a reduction of this subgroup to the subgroup $\begin{pmatrix} Gl(E_j) & O \\ O & H_j \end{pmatrix} \cap \begin{pmatrix} 1_E & O \\ O & G \end{pmatrix}$.

Since $P \times \mathbf{H}_j$ is a subbundle in $P \times \mathbf{G}$, there results that:

$$\Gamma(\pi^* TB) \cap (P \times \mathbf{H}_j) = \{0\} \qquad \forall \Gamma \in \gamma .$$

So, we shall consider the mapping $h_{\Gamma_j}: \Gamma \to h_{\Gamma_j}(\Gamma)$, where $h_{\Gamma_j}(\Gamma): \pi_j^* TP_j \to TP$ is given by:

$$h_{\Gamma_j}(\Gamma)(\pi_j^* TP_j) = \{\Gamma_j(\pi_j^* TP_j) \cap I(P \times \mathbf{G})\} \oplus \Gamma(\pi^* TB) .$$

We shall prove, finally, that the intersection from the above expression is a subvector bundle in TP over P. For this we consider the principal fibre bundles of linear frames of the vector bundles: TP, $\Gamma_j(\pi_j^* TP_j)$ and $I(P \times \mathbf{G})$; let them be $L(TP)$, $L(\Gamma_j(\pi_j^* TP)$, $L(I(P \times \mathbf{G})))$ and we prove that the intersection of the $Gl(E_j) \cap G$-fibre bundle $L(\Gamma_j(\pi_j^* TP_j)$ with the G-fibre bundle $L(I(P \times \mathbf{G}))$ is a subbundle in $L(TP)$. To this purpose we extend the results of D. Bernard from [1] related to the intersection of two subbundles of a differentiable principal finite dimensional fibre bundle to the banachic case.

Let G be a group manifold, ξ a principal fibre bundle with group G, G', G'' two closed group submanifolds in G, $\xi' = (P', G', \pi')$ and $\xi'' = (P'', G'', \pi'')$ two subbundle in ξ with groups G' and G'', respectively.

We reproduce the following two definitions from [1]:

DEFINITION 2. A continuous mapping f which maps the topological space Y into $G' \cdot G'' \subset G$ is called *local factorisable* in $G' \cdot G''$, if there exists an open cover $\{Y_\alpha\}$ of Y and the families of the continuous mappings $\{g'_\alpha\}$, $\{g''_\alpha\}$ which map Y_α into G'', respectively G', so that $f(y) = g'_\alpha(y) \cdot g''_\alpha(y)$, $\forall y \in Y_\alpha$.

DEFINITION 3. A pair of closed subgroup of the topological group G is called *regular* if every continuous mapping which maps a topological space Y into $G' \cdot G'' \subset G$, is local factorisable.

The following two results are proved in [1]:

I. Let G', G'' be two subgroup in G so that $G \to G/G'$, $G \to G/G''$ are the fiber bundles. If this pair is regular in G then the subgroup $G' \cap G''$ satisfies the condition that the projection $G \to G/G' \cap G''$ is a fibre bundle. For the pair to be regular, this condition is sufficient in one of the following cases:

1) G' (or G'') is open in G;
2) $G'/G' \cap G''$ (or $G''/G' \cap G''$) is compact.

II. $\xi' \cap \xi''$ is a subbundle of the topological principal fibre bundle ξ if and only if the following condition is satisfied: for a fixed cover of the base space B, $\{U_\alpha\}$, and for two local cross-sections over U_α in ξ', and ξ'' respectively, let them be $\sigma'_\alpha, \sigma''_\alpha$, so that $\sigma'_\alpha(x) = \sigma''_\alpha(x) \cdot g_\alpha(x)$, the mapping g_α maps U_α into $G' \cdot G''$ and is local factorisable.

From the result II there follows:

III. $\xi' \cap \xi''$ is a subbundle of the topological principal fibre bundle ξ if and only if the pair G', G'' is regular.

We apply these results to the bundles $L(TP)$, $L(\Gamma_j(\pi_j^* TP_j))$, $L(I(P \times \mathbf{G}))$ and to the pair $(Gl(E_j) \cap G, G)$. So, the condition 1 from I is satisfied and, moreover, $Gl(E_j) \cap G$ is a closed group submanifold in G; consequently, $G \to G/Gl(E_j) \cap G$ is a fiber bundles. From these things, there results that the pair $(Gl(E_j) \cap G, G)$ is regular and therefore the intersection $L(\Gamma_j(\pi_j^* TP_j)) \cap L(I(P \times \mathbf{G}))$ is a subbundle of the topological principal fibre bundle $L(TP)$. We shall prove, moreover, that this intersection is a subbundle of the *differential* principal fibre bundle $L(TP)$.

To this purpose, adding the condition of differentiation for the mappings which occur in the Definitions 2 and 3, we obtain, respectively, the notions of differentiable local factorisable mapping and of the generic pair of group submanifolds of a group manifold G.

From III and from the construction of the principal fibre bundles through the cocycles ([2], 6.4.3), we obtain:

LEMMA 1. *$\xi' \cap \xi''$ is a subbundle of the differentiable principal fibre bundle ξ if and only if the pair G', G'' is generic.*

In the finite dimensional case, this result is contained in the Proposition 1.6.2, [2], p. 175.

To have the possibility of stating a result which gives the sufficient condition for the pair G', G'' to be generic, we reproduce the following notations from [1]:

Let $q : G' \to G'/G' \cap G''$ be the principal fibre bundle, $p : G''(G') \to$ $\to G'/G' \cap G''$ the fibre bundle associated to q with G' as type fibre, $\alpha : G'' \times G' \to G''(G')$ the natural mapping, $\pi : G \to G/G''$ the principal fibre bundle, $f : G''(G') \to G$ given by: $f(\alpha(g'', g')) = g' \cdot g''$, which is a bijection on $G' \cdot G''$, and $f' : G'/G' \cap G'' \to G/G''$ given by the following commutative diagram:

$$\begin{tikzcd}
 & G''(G') \arrow[r, "f"] \arrow[dd, "p"'] & G \arrow[dd, "\pi"] \\
G' \arrow[urr, "i"] \arrow[dr, "q"'] & & \\
 & G'/G' \cap G'' \arrow[r, "f'"] & G/G''
\end{tikzcd}$$

i being the injection of G' in G.

With this we can state and prove the following:

LEMMA 2. *If the mapping f defined above is a homeomorphism, the pair G', G'' is generic.*

We make the proof in four steps:

1) f being a bijection, it determines a structure of differentiable manifold on $G' \cdot G''$.

2) We prove that $G' \cdot G''$ is a quasi-submanifold in G; to this purpose we consider the mapping $f: G''(G') \to G$ and we prove that it satisfies the conditions from the following enonciation, which we find in [2] 5.8.3., p. 48 in a slightly altered form:

Let $f: X \to Y$ be a morphism of manifolds. We suppose that f has the following property: the linear mapping $T_a(f)$ is injective and its image is a closed linear subspace in $T_{f(a)} Y$, $\forall a \in X$. Moreover, we suppose that f induces a homeomorphism of X into $f(X)$. Then $f(X)$ is a quasi-submanifold in Y.

First of all, we prove that df is an injective mapping in the point $f^{-1}(e)$, e being the neutral element in G.

Let be $x_0 = q(e)$, U an open neighbourhood of x_0, and let s be a differentiable cross-section of q over U, so that $s(x_0) = e$. Since $Tq \circ Ts = Id$, there results that $Ts(Tx_0 G'/G' \cap G'') = T_e G'$ and it is complementary with the tangent space $T_e G' \cap G''$. Moreover,

$$Ts(T_{x_0} G'/G' \cap G'') \cap T_e G'' \subset T_e G' \cap T_e G'' = T_e G' \cap G'',$$

and hence

$$Ts(T_{x_0} G'/G' \cap G'') \cap T_e G'' \subset Ts(T_{x_0} G'/G' \cap G'') \cap T_e G' \cap G'' = 0;$$

from this:

$$(1) \qquad Ts(T_{x_0} G'/G' \cap G'') \cap T_e G'' = 0.$$

The expression of the mapping f for a local map

$$\Phi: U \times G'' \to G''(G')$$

$$(x, g'') \to \alpha(g'', s(x))$$

of the space $G''(G')$, associated to the cross-section s, is the following: $f \circ \Phi(x, g'') = s(x) \cdot g''$. From this, there results that:

$$Tf(T_{\Phi(x_0, e)} G''(G')) = Tf(T_f - 1_{(e)} G''(G')) = Ts(T_{x_0} G'/G' \cap G'') + T_e G''.$$

From (1) there results that, the sum from the above expression is a direct sum; hence dl is a direct sum; hence df is injective in $f^{-1}(e)$. Moreover, since $T_e G' = Ts(T_{x_0} G'/G' \cap G'') + T_e G' \cap G''$, and $T_e G' \cap G'' \subset T_e G''$, there results that $Tf(T_f - l G''(G')) = T_e G' + T_e G''$ and therefore this is a closed linear subspace of $T_e G$.

The reasoning from [1] can be transposed without any difficulty to prove that f satisfies the above conditions in every point of the manifold $G''(G')$. Therefore there results that the range of the mapping f, which is the product $G' \cdot G''$, is a quasi-submanifold in G.

3) We prove that every mapping in G (with its range $G'\cdot G''$) differentiable in G, is differentiable in $G'\cdot G''$. To this purpose, we apply the following result from [2] 5.8.5, p. 48, to the real quasi-submanifold $G'\cdot G''$.

Let X be a quasi-submanifold of a manifold Y, and let g be a mapping of Z into X. We suppose that the field K has the characteristic null, or that X is a submanifold in Y. The mapping g is a morphism of Z in X if and only if g is a morphism of Z in Y.

4) We prove that every differentiable mapping in $G'\cdot G''$ is differentiable local factorisable, and therefore the pair G', G'' is generic.

For this, the reasoning from [1], Proposition 1.6.3, can be entirely transposed. With this, Lemma 2 is completely proved.

Now, we remark that f is a homeomorphism if and only if f' is a homeomorphism, and this last fact is satisfied, in particular, if G' or G'' are open in G.

Because the pair $(\mathrm{GL}(E_j) \cap G, G)$ satisfies this condition, there results that it is generic, and therefore the intersection $L(\Gamma_j(\pi_j^* TP_j)) \cap L(I(P \times \mathbf{G}))$ is a subbundle in the differential principal fibre bundle $L(TP)$.

Finally, we remark that the intersection $\Gamma_j(\pi_j^* TP_j) \cap I(P \times \mathbf{G})$ is the vector bundle associated to the previous subbundle, with $\mathbf{G} \cap T_{x_j} P_j (\forall x_j \in P_j)$ as type fibre and Theorem 2 is proved.

Examples:

I. Let E be a real Banach space with a flag, namely a double chain of the closed linear subspaces of E:

$$E_1 \subset E_2 \subset \cdots \subset E_n \cdots; \; E^{\infty-1} \supset E^{\infty-2} \supset \cdots \supset E^{\infty-n} \supset \cdots$$

so that:

1) $\dim E_j = j$; $\operatorname{codim} E^{\infty-i} = i$

2) $E_i \oplus E^{\infty-i} \simeq E \quad \forall i$

3) $\bigcup_i E_i$ is dense in E.

It is clear that a Banach space with Schauder basis admits a flag. With the notations from [7], let $Gl(n) = Gl(E, E_n; E^{\infty-n})$ be the subgroup of $Gl(E)$ of the linear inversible operators on E that apply E_n in itself and are the identity on $E^{\infty-n}$. It is obvious that $Gl(n)$ is closed and its Lie algebra, $L(n)$, admits the topological complement in $L(E)$, as his dimension is finite.

Let $\xi = (P, Gl(E), B, \pi)$ be the bundle of linear frames of a manifold B, modelled on E and $\mathfrak{N} = (Gl(E) \supset Gl(n) \supset \{1_E\})$ a sequence of closed group submanifolds in $Gl(E)$.

A refinement of ξ is given by the triplet $(\xi_{02}, \xi_{01}, \xi_{12})$ where $\xi_{02} = \xi$, $\xi_{01} = (P/Gl(n), Gl(E)/Dl(n), B, Gl(E)/Gl(n), \pi_{01})$ with $Dl(n) = \{\rho 1_{E_n}, \forall \rho \in R\}$ the diagonal subgroup, the largest normal subgroup in $Gl(E)$ contained in $Gl(n)$, and $\xi_{12} = (P, Gl(n), P/Gl(n), \pi_{12})$ a principal fibre bundle with group $Gl(n)$.

We suppose that there exists a global cross-section σ_{01} of the fibre bundle ξ_{01} and identify the manifold $\sigma_{01}(B)$ with B by means of the diffeomorphism $\pi_{01}|_{\sigma_{01}(B)}$. The principal fibre bundle $\xi_{12}|_{\sigma_{01}(B)}$ is a reduction of ξ to the subgroup $Gl(n)$, hence a $Gl(n)$-structure on B. This is given by a n-dimensional distribution on B, namely a differential system of the first order on B. There results that the refinements considered here, makes thus possible for us to investigate the properties of the finite dimensional distributions on Banach manifolds.

II. We consider now the subgroup $Gl(\infty - n) = Gl(E, E^{\infty-n}; E_n)$ of the continuous linear operators on E which apply $E^{\infty-n}$ in itself and are the identity on E_n. As its Lie algebra has the finite codimension in $L(E)$, there results that $Gl(\infty - n)$ is a group submanifold in $Gl(E)$.

Let ξ be the principal fibre bundle from Ex. I, and $\mathfrak{N}' = (Gl(E) \supset Gl(\infty - n) \supset \{1_E\})$. A refinement of ξ through $\mathfrak{N}'$ is given by: $(\xi_{02}, \xi'_{01}, \xi'_{12})$ where $\xi'_{01} = (P/Gl(\infty - n), Gl(E)/Dl(\infty - n) B, Gl(E)/Gl(\infty - n, \pi'_{01}),$ $Dl(\infty - n)) = \{\rho 1_{E\, \infty - n}, \forall \rho \in R\}$, $\xi'_{12} = (P, Gl(\infty - n) P/Gl(\infty - n), \pi'_{12})$ the last being a principal fibre bundle with the group $Gl(\infty - n)$.

By a reasoning analogous to that from Ex. I, we obtain a reduction of ξ at $Gl(\infty - n)$, namely a $Gl(\infty - n)$-structure given by a finite codimensional distribution on a Banach manifold B.

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