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Geodetic applications of conformal transformations of two-dimensional and three-dimensional harmonic vector fields

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Geodesia. — *Geodetic applications of conformal transformations of two-dimensional and three-dimensional harmonic vector fields.* Nota di FRANCO BOCCHIO, presentata (*) dal Socio A. MARUSSI.

RIASSUNTO. — Si studia la rappresentazione conforme che trasforma in geodetiche le linee di flusso e le linee equipotenziali di un campo vettoriale armonico su una superficie; l'estensione al caso tridimensionale permette di dedurre alcuni teoremi integrali che legano fra loro le curvature delle superfici equipotenziali nello spazio originario ed in quello trasformato.

1. The first part of the present paper is concerned with the application of conformal transformations to two-dimensional harmonic fields. It is shown that with a suitable choice of the scale factor of the transformation the streamlines and the equipotential lines of the field both transform into geodesics of the transformed surface. A similar problem was briefly taken into account by Lamb [1] in connection with the representation " in plano " of the irrotational solenoidal motion of a liquid in a curved stratum whose thickness is small compared with the radii of curvature.

The second part is concerned with a generalization to three dimensions of conformal transformations. Some differential relations between the curvature parameters of the equipotential surfaces in the original and in the transformed riemannian space are deduced which lead to some significant integral relations. Our interest is mainly devoted to harmonic vector fields of geodetic interest, i.e. to the gravitational field outside attracting masses.

2. Let **U** be a two-dimensional vector field given on a domain Σ of a regular surface. The " vorticity " of the field [2] is given by

$$\omega = \varepsilon^{ji} U_{i/j}, \qquad (i, j = 1, 2). \tag{2.1}$$

We can write

$$U_i = U\lambda_i \quad , \quad U = |\mathbf{U}| \tag{2.2}$$

λ_i being the unit tangent to the field lines. It follows

$$\omega = \varepsilon^{ji} (U\lambda_i)_{/j} = U\varepsilon^{ji} \lambda_{i/j} - \nu^j U_{/j} \tag{2.3}$$

ν^j being the unit normal to the streamlines.

Remembering that

$$\nu^j{}_{/j} = \varepsilon^{ij} \lambda_{i/j} = - \varepsilon^{ji} \lambda_{i/j} \tag{2.4}$$

(*) Nella seduta del 9 dicembre 1972.

and

$$\gamma = -\nu^j{}_{/j} \tag{2.5}$$

γ being the geodesic curvature, we obtain

$$\omega = U\gamma - \nu^j U_{/j} . \tag{2.6}$$

3. If the vector field is irrotational, $U_i = \varphi_{/i}$,

$$\gamma = \frac{U_{/j} \nu^j}{U} \tag{2.7}$$

or

$$\gamma = \mu_{/j} \nu^j \quad , \quad \mu = \ln U \tag{2.8}$$

and the geodesic curvature γ^* of the equipotential lines is given by Beltrami's formula

$$\gamma^* = -\frac{\Delta_2 \varphi}{\sqrt{\Delta_1 \varphi}} - \Delta_1 \left(\varphi , \frac{1}{\sqrt{\Delta_1 \varphi}}\right) \tag{3.2}$$

where

$$\begin{cases} \Delta_1 \varphi = g^{ij} \varphi_{/i} \varphi_{/j} \\ \Delta_1 (\psi_1 , \psi_2) = g^{ij} \psi_{1/i} \psi_{2/j} \\ \Delta_2 \varphi = g^{ij} \varphi_{/ij} \end{cases}$$

and g_{ij} is the metric tensor. If furthermore the field is solenoidal, we have

$$\gamma^* = -\Delta_1 \left(\varphi , \frac{1}{U}\right) = \mu_{/i} \nu^{*i} \tag{3.3}$$

$\nu^{*i} = \lambda^i$ being the normal to the equipotential lines.

4. Under a conformal transformation with scale factor

$$e^\mu = |\text{grad } \varphi| = U \tag{4.1}$$

the streamlines and the equipotential lines of the field both transform into geodesics of $\bar{\Sigma}$. In fact, by Schol's theorem we have

$$\bar{\gamma} = \bar{e}^\mu (\gamma - \mu_{/i} \nu^i) \tag{4.2}$$

which gives the transformed geodesic curvature of a surface curve with unit normal ν^i.

5. The condition $\Delta_2 \varphi = 0$ implies [5] that the streamlines and the equipotential lines form an isometric system with

$$ds^2 = \frac{1}{\Delta_1 \varphi} (d\varphi^2 + d\psi^2) \tag{5.1}$$

the lines $\psi =$ const. being the streamlines and $\Delta_1 \varphi = \Delta_1 \psi$. It can be shown that the transformed surface $\bar{\Sigma}$ is a plane since from $e^{\mu} = \sqrt{\Delta_1 \varphi}$ we obtain for $d\bar{s}^2$ the pythagorean form

$$d\bar{s}^2 = e^{2\mu}\, ds^2 = d\varphi^2 + d\psi^2. \tag{5.2}$$

6. We now focus our attention on the extension of these results to the three-dimensional space. If η^i is the unit normal to the surface $\varphi =$ const. embedded in the Euclidean space, we have by covariant differentiation

$$\varphi_{/ij} = (U\eta_i)_{/j} = U_{/j}\,\eta_i + U\eta_{i/j} = U_{/i}\,\eta_j + U\eta_{j/i} \qquad (i\,,j = 1\,,2\,,3) \tag{6.1}$$

since $\varphi_{/ij} = \varphi_{/ji}$ and therefore

$$\eta_{i/j} + \mu_{/j}\,\eta_i = \eta_{j/i} + \mu_{/i}\,\eta_j \quad , \quad \mu = \ln U. \tag{6.2}$$

By multiplication with η^i it follows

$$\eta^i\,\eta_{i/j} + \mu_{/j} = \eta^i\,\eta_{j/i} + \mu_{/i}\,\eta^i\,\eta_j\,. \tag{6.3}$$

Since

$$\begin{cases} \eta^i\,\eta_{i/j} = 0 \\ \eta^i\,\eta_{j/i} = \chi\nu_j \end{cases} \tag{6.4}$$

in which χ is the first curvature of the streamlines and ν^i their principal normal,

$$\chi = \mu_{/j}\,\nu^j. \tag{6.5}$$

Equation (6.5) suggests at once that a conformal transformation with scale factor $e^{\mu} = |\text{grad}\,\varphi|$ transforms the streamlines into geodesics of $\bar{s}_3$. The above result has been widely applied to a number of different physical problems of geodetic interest by Hotine [6], Marussi [7], [8] and Bocchio [3], [4]. If the field is furthermore harmonic we have

$$\mu_{/i}\,\eta^i = M. \tag{6.6}$$

7. The mean curvature $\bar{M}$ and the "relative" total curvature $\bar{K}$ [8] of the transformed equipotential surfaces are given by

$$\bar{M} = \bar{e}^{\mu}\,(M - 2\,\mu_{/i}\,\eta^i) \tag{7.1}$$

$$\bar{K} = \bar{e}^{2\mu}\,[K - M\mu_{/i}\,\eta^i + (\mu_{/i}\,\eta^i)^2]\,, \tag{7.2}$$

or

$$\bar{M} = -\,\bar{e}^{\mu}\,M \tag{7.3}$$

$$\bar{K} = \bar{e}^{2\mu}\,K. \tag{7.4}$$

It follows that the integral mean curvature $\int_L M\, dl$ along any line, apart of the sign, and the integral total " relative " curvature $\iint_\Sigma K\, d\sigma$ are invariant for this transformation.

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