Atti Accademia Nazionale dei Lincei

Classe Scienze Fisiche Matematiche Naturali

Rendiconti

Udai Pratad Singh

On the induced theory of Finsler hypersurfaces from the standpoint of non-linear connections

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Serie 8, Vol. **53** (1972), n.6, p. 541–548.

Accademia Nazionale dei Lincei

<http://www.bdim.eu/item?id=RLINA_1972_8_53_6_541_0>

Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti, Accademia Nazionale dei Lincei, 1972.

Geometria differenziale. — *On the induced theory of Finsler hypersurfaces from the standpoint of non-linear connections.* Nota di

UDAI PRATAD SINGH, presentata (*) dal Socio E. BOMPIANI.

RIASSUNTO. — Le connessioni non lineari negli spazi di Finsler sono state studiate da Vagner, Barthel e Kawaguchi (v. bibliografia). Nella presente Nota si studiano le connessioni indotte in una ipersuperficie di Finsler. In particolare si danno le condizioni necessarie e sufficienti affinchè una connessione metrica (secondo Rund) nell'ambiente induca una connessione pure metrica nell'ipersuperficie. Si studiano anche relazioni fra le geodetiche di una ipersuperficie e quelle dell'ambiente d'immersione.

1. INTRODUCTION

We outline below some fundamental formulae which will be used in the subsequent sections of this paper.

Let X^i be a vectorfield, $g_{ij}(x, X)$ be the components of the metric tensor of a Finsler space F_n and $Y_i = g_{ij}(x, X) X^j$. Suppose we are given functions $\overset{1}{\Gamma}{}^i_k(x, X)$ and $\overset{2}{\Gamma}_{ik}(x, Y)$ such that the absolute differentials

$$\overset{1}{\delta} X^i = dX^i + \overset{1}{\Gamma}{}^i_k(x, X)\, dx^k \tag{1.1}$$

and

$$\overset{2}{\delta} Y_i = dY_i - \overset{2}{\Gamma}_{ik}(x, Y)\, dx^k \tag{1.2}$$

are respectively the components of contravariant and covariant vectors. The functions $\overset{1}{\Gamma}{}^k_i(x, X)$, $\overset{2}{\Gamma}_{ik}(x, Y)$ are supposed to be positively homogeneous of first degree in X and Y respectively. These are used in defining the connection parameters

$$\overset{1}{\Gamma}{}^i_{jk}(x, X) = \frac{\partial \overset{1}{\Gamma}{}^i_k}{\partial X^j} \quad , \quad \overset{2}{\Gamma}{}^i_{jk}(x, Y) = \frac{\partial \overset{2}{\Gamma}_{jk}(x, Y)}{\partial Y_i}\,. \tag{1.3}$$

We mention the following two conditions:

(A) If X^i undergoes parallel displacement (i.e. $\overset{1}{\delta} X^i = 0$) then so does Y_i (i.e. $\overset{2}{\delta} Y_i = 0$). This condition is characterised by (Rund [4], page 238)

$$\overset{2}{\Gamma}_{ik}(x, Y) = \frac{\partial g_{ij}(x, X)}{\partial x^k} X^j - g_{ij}\, \overset{1}{\Gamma}{}^j_k(x, X)\,. \tag{1.4}$$

(*) Nella seduta del 9 dicembre 1972.

(B) The connection defined by $\overset{1}{\Gamma}{}^i_k(x, X)$ is metric, i.e. the length of the vectorfield X^i remains unchanged under parallel displacement. In other words

$$\overset{1}{\delta}(g_{ij}(x, X)\, X^i X^j) = 0 \qquad \text{for} \quad \overset{1}{\delta}X^i = 0$$

which yields

$$\overset{1}{\delta}g_{ij}(x, X)\, X^i X^j = 0 . \tag{1.5}$$

This condition is characterised by ([4] page 239)

$$Y_i \overset{1}{\Gamma}{}^i_k(x, X) = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} X^i X^j. \tag{1.6}$$

2. Induced connection parameters

Let $F_{n-1} : x^i = x^i(u^\alpha)$; $i = 1, \cdots, n$; $\alpha = 1, \cdots, n-1$ be a hypersurface of F_n. The components X^i, X^α of a vectorfield of the hypersurface are related by

$$X^i = B^i_\alpha X^\alpha \qquad \text{where} \quad B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha} \cdot \tag{2.1}$$

The induced differential $\overset{1}{\delta}X^\alpha$ is defined by

$$\overset{1}{\delta}X^\alpha = B^\alpha_i \overset{1}{\delta}X^i, \tag{2.2}$$

where $B^\alpha_i = g^{\alpha\beta}(u, X)\, g_{ij}(x, X)\, B^j_\beta$, $g_{\alpha\beta}(u, X)$ being the metric tensor of F_{n-1}. The equation (2.1) yields

$$dX^i = B^i_\beta\, dX^\beta + B^i_{\beta\gamma} X^\beta du^\gamma \qquad \left(B^i_{\beta\gamma} = \frac{\partial^2 x^i}{\partial u^\beta \partial u^\gamma}\right). \tag{2.3}$$

Defining

$$\overset{1}{\delta}X^\alpha = dX^\alpha + \overset{1}{\Gamma}{}^\alpha_\gamma(u, X)\, du^\gamma \tag{2.4}$$

and using (1.1), (2.2) and (2.3) we find

$$\overset{1}{\Gamma}{}^\alpha_\gamma = B^\alpha_i (B^i_{\beta\gamma} X^\beta + \overset{1}{\Gamma}{}^i_k B^k_\gamma), \tag{2.5}$$

where we have used the relations $B^\alpha_i B^i_\beta = \delta^\alpha_\beta$, $dx^k = B^k_\gamma du^\gamma$.

After putting

$$Y_i = g_{ij}(x, X)\, X^j \quad , \quad Y_\alpha = g_{\alpha\beta}(u, X)\, X^\beta \tag{2.6}$$

and using (2.1) we obtain

$$Y_i = B_i^\alpha \, Y_\alpha . \tag{2.7}$$

We now define another induced differential

$$\overset{2}{\delta} Y_\alpha = B_\alpha^i \, \overset{2}{\delta} Y_i . \tag{2.8}$$

The differentiation of (2.6) gives

$$dY_i = \frac{\partial g_{ij}(x, X)}{\partial x^k} X^j \, dx^k + g_{ij}(x, X)\,(B_{\beta\gamma}^j \, X^\beta \, du^\gamma + B_\beta^j \, dX^\beta) \tag{2.9}$$

and

$$dY_\alpha = \frac{\partial g_{\alpha\beta}(u, X)}{\partial u^\gamma} X^\beta \, du^\gamma + g_{\alpha\beta}(u, X)\, dX^\beta . \tag{2.10}$$

Defining

$$\overset{2}{\delta} Y_\alpha = dY_\alpha - \overset{2}{\Gamma}_{\alpha\gamma} \, du^\gamma$$

and simplifying with the help of (1.2), (2.9), (2.10) and the relation obtained after differentiation (with respect to u^γ) of

$$g_{\alpha\beta}(u, X) = g_{ij}(x, X)\, B_\alpha^i \, B_\beta^j$$

we get

$$\overset{2}{\Gamma}_{\beta\gamma}(u, Y) = (Y_j \, B_{\beta\gamma}^j + \overset{2}{\Gamma}_{hk}(x, Y)\, B_\beta^h \, B_\gamma^k) . \tag{2.11}$$

It is assumed that the function $\overset{1}{\Gamma}{}_\gamma^\alpha(u, X)$ and $\overset{2}{\Gamma}_{\beta\gamma}(u, Y)$ are differentiable. We now define

$$\overset{1}{\Gamma}{}_{\beta\gamma}^\alpha = \frac{\partial \overset{1}{\Gamma}{}_\gamma^\alpha}{\partial X^\beta} \quad , \quad \overset{2}{\Gamma}{}_{\beta\gamma}^\alpha = \frac{\partial \overset{2}{\Gamma}_{\beta\gamma}}{\partial Y_\alpha} . \tag{2.12}$$

A direct differentiation of the relation

$$B_i^\alpha = g^{\alpha\delta}(u, X)\, g_{ij}(x, X)\, B_\delta^j$$

with respect to X^β will yield (after some simplification)

$$\frac{\partial B_i^\alpha}{\partial X^\beta} = 2\, N_i \, M_\beta^\alpha , \tag{2.13}$$

where we have used the fact

$$B_i^\alpha \, B_\alpha^j = (\delta_i^j - N^j \, N_i) , \tag{2.14}$$

N_i being the covariant components of the unit normal vector,

$$2C_{ijk}(x, X) = \frac{\partial g_{ij}(x, X)}{\partial X^k},$$

$$M_{\alpha\beta}(u, X) = C_{ijk}(x, X)\, B^i_\alpha\, B^j_\beta\, N^k \quad \text{and} \quad M^\alpha_\beta(u, X) = g^{\alpha\gamma} M_{\beta\gamma}(u, X).$$

Differentiating (2.5) with respect to X^β, using the relations (1.3), (2.12), (2.13) and the fact $M^\alpha_\beta(u, X)\, X^\beta = 0$, we find

$$\overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u, X) = 2N_i\, \overset{1}{\Gamma}{}^i_k(x, X)\, B^k_\gamma\, M^\alpha_\beta + B^\alpha_i\,(B^i_{\beta\gamma} + \overset{1}{\Gamma}{}^i_{hk}\, B^h_\beta\, B^k_\gamma) + \\ + 2N_i\, M^\alpha_\beta\, B^i_{\delta\gamma}\, X^\delta. \tag{2.15}$$

In order to evaluate $\overset{2}{\Gamma}{}^\alpha_{\beta\gamma}(u, X)$, we notice that a direct differentiation of $Y_i = B^\alpha_i Y_\alpha$ will give

$$\frac{\partial Y_i}{\partial Y_\beta} = B^\beta_i, \tag{2.16}$$

where we have used (2.13) and the relations $\dfrac{\partial X^\gamma}{\partial Y_\beta} = g^{\gamma\beta}$, $M^\alpha_\gamma(u, X)\, Y_\alpha = 0$ in the simplification. Differentiating (2.11) with respect to Y_α and using (1.3), (2.12), (2.16) we obtain

$$\overset{2}{\Gamma}{}^\alpha_{\beta\gamma}(u, Y) = B^\alpha_i\,(B^i_{\beta\gamma} + \overset{2}{\Gamma}{}^i_{hk}(x, Y)\, B^h_\beta\, B^k_\gamma). \tag{2.17}$$

The connection parameters $\overset{1}{\Gamma}{}^\alpha_{\beta\gamma}(u, X)$, $\overset{2}{\Gamma}{}^\alpha_{\beta\gamma}(u, Y)$ are non-symmetric in β, γ and positively homogeneous of zero degree in X, Y respectively. These will be called induced "non-linear connection parameters" of the hypersurface.

3. Properties of induced non-linear connections

Consider the following conditions in F_n and F_{n-1} respectively.

(A_1). If X^i undergoes parallel displacement in F_n then so does Y_i.

(A_2). If X^α undergoes parallel displacement in F_{n-1} then so does Y_α.

The condition (A_1) is characterised by (1.4) and the condition (A_2) is characterised by the corresponding relation

$$\overset{2}{\Gamma}_{\alpha\gamma}(u, X) = \frac{\partial g_{\alpha\beta}(u, X)}{\partial u^\gamma} X^\beta - g_{\alpha\beta}(u, X)\, \overset{1}{\Gamma}{}^\beta_\gamma(u, X) \tag{3.1}$$

in the space F_{n-1}. We shall prove the following:

THEOREM 3.1. *A necessary and sufficient condition that* (A_2) *holds in the hypersurface is that* (A_1) *holds in the enveloping space.*

Proof. The differentiation of

$$g_{\alpha\beta}(x\,,\,X) = g_{ij}(x\,,\,X)\,B^i_\alpha\,B^j_\beta$$

gives

$$\text{(3.2)}\qquad Y_i\,B^i_{\alpha\gamma} + g_{ij}\,B^i_{\delta\gamma}\,B^j_\alpha\,X^\delta - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\,X^\beta = \frac{\partial g_{ij}}{\partial x^k}\,B^i_\alpha\,B^k_\gamma\,X^j.$$

A simple calculation based on the equations (2.5), (2.11) and (3.2) will yield

$$\text{(3.3)}\qquad \overset{2}{\Gamma}_{\alpha\gamma} - \frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\,X^\beta + g_{\alpha\beta}\,\overset{1}{\Gamma}{}^\beta_\gamma = \left(\overset{2}{\Gamma}_{ik} - \frac{\partial g_{ij}}{\partial x^k}\,X^j + g_{ij}\,\overset{1}{\Gamma}{}^j_k\right) B^i_\alpha\,B^k_\gamma\,,$$

where we have used the fact $g_{\alpha\beta}\,B^\beta_i = g_{ij}\,B^j_\alpha$. Since the above relation is true for every α and γ, therefore condition (3.1) implies and is implied by condition (1.4). This proves the theorem.

Let us now consider the conditions:

(B_1). $\overset{1}{\Gamma}{}^i_j(x\,,\,X)$ is a metric connection in F_n.

(B_2). $\overset{1}{\Gamma}{}^\alpha_\beta(x\,,\,X)$ is a metric connection in F_{n-1}.

The condition (B1) is characterised by (1.6) while the condition (B2) is characterised by

$$\text{(3.4)}\qquad Y_\alpha\,\overset{1}{\Gamma}{}^\alpha_\gamma(u\,,\,X) = \frac{1}{2}\,\frac{\partial g_{\alpha\beta}(u\,,\,X)}{\partial u^\gamma}\,X^\alpha\,X^\beta.$$

We shall prove the following:

THEOREM 3.2. *A necessary and sufficient condition that* (B_2) *holds in* F_{n-1} *is that* (B1) *holds in* F_n.

Proof. Substituting from the equation (2.5) and (3.2) and using (2.7) in the simplification we find

$$\text{(3.5)}\qquad \left(Y_\alpha\,\overset{1}{\Gamma}{}^\alpha_\gamma - \frac{1}{2}\,\frac{\partial g_{\alpha\beta}}{\partial u^\gamma}\,X^\alpha\,X^\beta\right) = \left(Y_i\,\overset{1}{\Gamma}{}^i_k - \frac{1}{2}\,\frac{\partial g_{ij}}{\partial x^k}\,X^i\,X^j\right) B^k_\gamma\,.$$

The theorem follows from (1.6), (3.4) and the fact that (3.5) is true for every γ.

4. GEODESICS IN THE HYPERSURFACE

The geodesics of F_n and F_{n-1} are given by (Rund [4], page 240)

$$\text{(4.1)}\qquad \frac{\overset{1}{\delta}X^i}{\delta s} + g^{ih}\,Y_j\,(\overset{1}{\Gamma}{}^j_{hk}\,X^k - \overset{1}{\Gamma}{}^j_h) = 0$$

and

$$\text{(4.2)}\qquad \frac{\overset{1}{\delta}X^\alpha}{\delta s} + g^{\alpha\gamma}\,Y_\beta\,(\overset{1}{\Gamma}{}^\beta_{\gamma\delta}\,X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) = 0\,.$$

Equations (2.2) and (2.14) will yield

$$\frac{\overset{1}{\delta} X^i}{\delta s} = B^i_\alpha \frac{\overset{1}{\delta} X^\alpha}{\delta s} + N^i N_j \frac{\overset{1}{\delta} X^j}{\delta s} \,. \tag{4.3}$$

A calculation based on equations (2.15), (2.5), (1.6) and relations (Rund [4] page 236)

$$\overset{1}{\Gamma}{}^i_{hk} X^h = \overset{1}{\Gamma}{}^i_k \quad , \quad M^\beta_\gamma Y_\beta = 0 \qquad \text{and} \qquad g^{\alpha\gamma} B^i_\alpha B^h_\gamma = g^{ih} - N^i N^h$$

gives

$$g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) B^i_\alpha = \tag{4.4}$$

$$= g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) - N^i N^h Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) \,.$$

Further after putting $X^i = \frac{dx^i}{ds}$, $X^\alpha = \frac{du^\alpha}{ds}$ and substituting

$$\frac{dX^j}{ds} = B^j_{\beta\gamma} X^\beta X^\gamma + B^j_\beta \frac{dX^\beta}{ds} \quad , \quad \overset{1}{\Gamma}{}^j_k \frac{dx^k}{ds} = \overset{1}{\Gamma}{}^j_{hk} B^h_\beta B^k_\gamma X^\beta X^\gamma$$

in

$$N_j \frac{\overset{1}{\delta} X^j}{\delta s} = N_j \left(\frac{dX^j}{ds} + \overset{1}{\Gamma}{}^j_k \frac{dx^k}{ds} \right) \tag{4.5}$$

we find

$$N_j \frac{\overset{1}{\delta} X^j}{\delta s} = \overline{\Omega}_{\beta\gamma} (u \,,\, X) X^\beta X^\gamma \tag{4.6}$$

where

$$\overline{\Omega}_{\beta\gamma} (u \,,\, X) = N_j (B^j_{\beta\gamma} + \overset{1}{\Gamma}{}^j_{hk} (x \,,\, X) B^h_\beta B^k_\gamma) \,.$$

The tensor with the components $\overline{\Omega}_{\beta\gamma} (u \,,\, X)$ is called second fundamental tensor of the hypersurface. It is obviously a non-symmetric tensor.

Further in view of the fact $\overset{1}{\Gamma}{}^j_h (x \,,\, X) = \overset{1}{\Gamma}{}^j_{kh} (x \,,\, X) X^k$ we find

$$N^h Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) = \hat{\Omega}_{\beta\gamma} (u \,,\, X) X^\beta X^\gamma \tag{4.7}$$

where

$$\hat{\Omega}_{\beta\gamma} (u \,,\, X) = N^h g_{jl} (\overset{1}{\Gamma}{}^j_{hk} - \overset{1}{\Gamma}{}^j_{kh}) B^l_\beta B^k_\gamma \,.$$

Substituting from (4.4), (4.6) in (4.3) and simplifying with the help of (4.7) we get

$$\frac{\overset{1}{\delta} X^i}{\delta s} + g^{ih} Y_j (\overset{1}{\Gamma}{}^j_{hk} X^k - \overset{1}{\Gamma}{}^j_h) = B^i_\alpha \left[\frac{\overset{1}{\delta} X^\alpha}{\delta s} + g^{\alpha\gamma} Y_\beta (\overset{1}{\Gamma}{}^\beta_{\gamma\delta} X^\delta - \overset{1}{\Gamma}{}^\beta_\gamma) \right] + \tag{4.8}$$

$$+ \Delta_{\beta\gamma} (u \,,\, X) X^\beta X^\gamma N^i$$

where

(4.9) $$\Delta_{\beta\gamma}(u\,,\,X) = \overline{\Omega}_{\beta\gamma}(u\,,\,X) + \hat{\Omega}_{\beta\gamma}(u\,,\,X)\,.$$

The scalar

(4.10) $$\underset{n}{k} = \Delta_{\beta\gamma}(u\,,\,X)\,X^{\beta}\,X^{\gamma}$$

will be called the normal curvature and X^{α} will be called along the asymptotic line if

(4.11) $$\Delta_{\beta\gamma}(u\,,\,X)\,X^{\beta}\,X^{\gamma} = 0\,.$$

The equations (4.1), (4.2), (4.8) and (4.11) may be used in proving the following:

THEOREM 4.1. *A geodesic of* F_n *is both a geodesic and an asymptotic line of the hypersurface. Conversely, a geodesic of the hypersurface is a geodesic of the enveloping space if and only if it is an asymptotic line.*

It may be noted that contrary to usual convention the normal curvature is not given by $\overline{\Omega}_{\beta\gamma}(u\,,\,X)\,X^{\beta}\,X^{\gamma}$. However, in view of equations (4.7), (4.9) and (4.10) this ($\underset{n}{k} = \overline{\Omega}_{\beta\gamma}(u\,,\,X)\,X^{\beta}\,X^{\gamma}$) will happen if the condition

(C$_1$) $$Y_j\,(\overset{1}{\Gamma}{}^{j}_{hk}\,X^{k} - \overset{1}{\Gamma}{}^{j}_{h}) = 0$$

is true. We shall prove

THEOREM 4.2. *The condition* (C$_1$) *holds in* F_n *if and only if the corresponding condition*

(C$_2$) $$Y_{\alpha}\,(\overset{1}{\Gamma}{}^{\alpha}_{\beta\gamma}\,X^{\gamma} - \overset{1}{\Gamma}{}^{\alpha}_{\beta}) = 0$$

holds in F_{n-1}.

Proof. A calculation based on the equations (2.15), (2.5) and the conditions $Y_{\alpha}\,M^{\alpha}_{\gamma} = 0\,,\,Y_{\alpha}\,B^{\alpha}_{i} = Y_i$ yields

$$Y_{\alpha}\,(\overset{1}{\Gamma}{}^{\alpha}_{\beta\gamma}\,X^{\gamma} - \overset{1}{\Gamma}{}^{\alpha}_{\beta}) = Y_i\,(\overset{1}{\Gamma}{}^{i}_{hk}\,X^{k} - \overset{1}{\Gamma}{}^{i}_{h})\,B^{h}_{\beta}\,.$$

Since this is true for every β, therefore (C$_1$) implies and is implied by (C$_2$). This proves the theorem.

It has been proved in [4] (page 240) that (C$_1$) is a necessary and sufficient condition in order that the geodesics of F_n may be auto-parallel curves. Theorem 4.2 may now be put in the form:

THEOREM 4.3. *The geodesics of* F_{n-1} *are auto-parallel curves if and only if the geodesics of* F_n *are auto-parallel.*

The following theorem is immediate from equations (4.9), (4.10), (4.11) and condition (C$_1$).

THEOREM 4.4. *The asymptotic lines of* F_{n-1} *are given by*

$$\overline{\Omega}_{\beta\gamma}(u\,,X)\,X^{\beta}\,X^{\gamma}=0$$

if and only if the geodesics of F_n *are auto-parallel.*

Let $C: u^{\alpha}=u^{\alpha}(s)$ be a curve of F_{n-1} and $X^{\alpha}=\frac{du^{\alpha}}{ds}$. The vectors

$$q^{i}=\frac{\overset{1}{\delta}X^{i}}{\delta s}+g^{ih}\,Y_{j}\,(\overset{1}{\Gamma}{}^{j}_{hk}\,X^{k}-\overset{1}{\Gamma}{}^{j}_{h})\quad,\quad p^{\alpha}=\frac{\overset{1}{\delta}X^{\alpha}}{\delta s}+g^{\alpha\gamma}\,Y_{\beta}\,(\overset{1}{\Gamma}{}^{\beta}_{\gamma\delta}\,X^{\delta}-\overset{1}{\Gamma}{}^{\beta}_{\gamma})$$

are called the first curvature vectors of the curve with respect to F_n and F_{n-1} respectively. The first curvature vector q^i is, in general, different from the derived vector $\overset{1}{\delta}X^{i}/\delta s$ of the unit tangent X^i. However, it is easy to prove

THEOREM 4.5. *The first curvature vector* q^i *differs from the derived vector* $\overset{1}{\delta}X^{i}/\delta s$ *by a vector which is orthogonal to the tangent vector* X^i.

Also we have

THEOREM 4.6. *The derived vector* $\overset{1}{\delta}X^{i}/\delta s$ *is orthogonal to* X^i *if and only if* $\overset{1}{\Gamma}{}^{i}_{k}(x\,,X)$ *is a metric connection.*

Proof. Differentiating $g_{ij}(x\,,X)\,X^i\,X^j=1$ we find

$$g_{ij}\,\frac{\overset{1}{\delta}X^{i}}{\delta s}\,X^{i}=-\frac{1}{2}\,\frac{\overset{1}{\delta}g_{ij}}{\delta s}\,X^{i}\,X^{j}. \tag{4.12}$$

The theorem is immediate from the equations (1.5) and (4.12).

Using Theorems 4.6 and 4.5 we have

THEOREM 4.7. *The first curvature vector* q^i *is orthogonal to* X^i *if and only if* $\overset{1}{\Gamma}{}^{i}_{j}(x\,,X)$ *is a metric connection.*

Finally, Theorems 3.2 and 4.7 yield

THEOREM 4.8. *The first curvature vector with respect to* F_{n-1} *is orthogonal to* X^{α} *if and only if the vector* q^i *(the first curvature vector with respect to* F_n*) is orthogonal to* X^i.

REFERENCES

[1] VAGNER V. V., *The geometry of Finsler as a theory of field of local hypersurfaces in* X_n, « Trudy Sem. Vektor. Tenzor. Analizu », 7, 65–166 (Russian 1949).

[2] BARTHEL W., *Ueber eine parallelverschiebung mit Längeninvarianz in lokal Minkowskische Raumen I, II*, « Arch. Math. », 4, 346–365 (1953).

[3] KAWAGUCHI A., *On the theory of non-linear connections*, « Tensor N.S. », 6, 165–199 (1956).

[4] RUND H., *The differential geometry of Finsler spaces*, Springer-Verlag (1959).