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4-gonal subconfigurations of a given 4-gonal configuration

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Geometria. — *4-gonal subconfigurations of a given 4-gonal configuration.* Nota di JOSEPH ADOLPHE THAS, presentata (*) dal Socio B. SEGRE.

RIASSUNTO. — Si introducono e studiano certe strutture finite, includenti quelle formate dai punti e dalle rette di una quadrica non degenere (su cui non giacciano piani) di uno spazio di Galois di dimensione 3, o 4, o 5.

1. INTRODUCTION

1.1. DEFINITION. A finite 4-gonal configuration [2] is an incidence structure $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, with an incidence relation satisfying the following axioms.

(i) each point is incident with r lines ($r \geq 2$) and two distinct points are incident with at most one line;

(ii) each line is incident with k points ($k \geq 2$) and two distinct lines are incident with at most one point;

(iii) if x is a point and L is a line not incident with x, then there are a unique point x' and a unique line L' such that $x \,\mathrm{I}\, L' \,\mathrm{I}\, x' \,\mathrm{I}\, L$.

1.2. FUNDAMENTAL RELATIONS. *If* $|\mathbf{P}| = v$ and $|\mathbf{B}| = b$, then $v = k(kr - k - r + 2)$ and $b = r(kr - k - r + 2)$. In [4] D. G. Higman proves that the positive integer $k + r - 2$ divides $kr(k-1)(r-1)$. Moreover, under the assumption that $k > 2$ and $r > 2$, he shows that $r - 1 \leq (k-1)^2$ and $k - 1 \leq (r-1)^2$.

1.3. EXAMPLES OF 4-GONAL CONFIGURATIONS. (a) Let $\mathbf{P} = \{x_{ij} \| i, j = 1, 2, \cdots, k\}$ and $\mathbf{B} = \{L_1, L_2, \cdots, L_k, M_1, M_2, \cdots, M_k\}$, where $k \geq 2$. Incidence is defined as follows: $x_{ij} \,\mathrm{I}\, L_l \Longleftrightarrow i = l$, $x_{ij} \,\mathrm{I}\, M_l \Longleftrightarrow j = l$. Then $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$ is a 4-gonal configuration with parameters $k = k$, $r = 2$, $v = k^2$, $b = 2k$. This 4-gonal configuration is denoted by $\mathbf{T}(k)$.

(b) We consider a non-singular hyperquadric Q of index 2 of the projective space $PG(d, q)$, with $d = 3, 4$ or 5. Then the points of Q together with the lines of Q (which are the subspaces of maximal dimension

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on Q) form a 4-gonal configuration $\mathbf{Q}(d, q)$ with parameters [2]

$$k = q + 1 \quad , \quad r = 2 \quad , \quad v = (q+1)^2 \quad , \quad b = 2(q+1), \quad \text{when } d = 3;$$

$$k = r = q + 1 \quad , \quad v = b = (q+1)(q^2+1), \quad \text{when } d = 4;$$

$$k = q + 1 \quad , \quad r = q^2 + 1 \quad , \quad v = (q+1)(q^3+1) \quad , \quad b = (q^2+1)(q^3+1), \quad \text{when } d = 5.$$

REMARKS. 1) $\mathbf{Q}(3, q)$ is isomorphic to $\mathbf{T}(q+1)$.

2) The points of $PG(3, q)$, together with the totally isotropic lines with respect to a symplectic polarity π, form a 4-gonal configuration $\mathbf{W}(q)$ which is isomorphic to the dual of $\mathbf{Q}(4, q)$ [1].

(c) Let H be a non-singular Hermitian primal [8] of the projective space $PG(d, q)$, $q = p^{2h}$. If $d = 3$ or 4, then the points of H together with the lines of H form a 4-gonal configuration $\mathbf{H}(d, q)$ with parameters [2]

$$k = q + 1 \quad , \quad r = 1 + \sqrt{q} \quad , \quad v = (1+q)(1+q\sqrt{q}) \quad , \quad b = (1+\sqrt{q})(1+q\sqrt{q}), \quad \text{when } d = 3;$$

$$k = q + 1 \quad , \quad r = 1 + q\sqrt{q} \quad , \quad v = (1+q)(1+q^2\sqrt{q}) \quad , \quad b = (1+q\sqrt{q})(1+q^2\sqrt{q}), \quad \text{when } d = 4.$$

(d) Consider an oval O (i.e. a set of $q + 2$ points no three of which are collinear) of the plane $PG(2, q)$, $q = 2^h$. Let $PG(2, q)$ be embedded as a plane H in $PG(3, q) = P$. Now a 4-gonal configuration $\mathbf{O}(q)$ is defined as follows [3]. Points of $\mathbf{O}(q)$ are the points of $P - H$. Lines of $\mathbf{O}(q)$ are the lines of P which are not contained in H and meet O (necessarily in a unique point). Incidence is that of P. The 4-gonal configuration $\mathbf{O}(q)$, $q = 2^h$, so defined has parameters

$$k = q, \quad r = q + 2, \quad v = q^3, \quad b = q^2(q+2).$$

2. OVALOIDS AND SPREADS

2.1. DEFINITIONS. An ovaloid (resp. spread) of the 4-gonal configuration $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, with parameters k, r, v, b, is a set of $kr - k - r + 2 = \theta$ points (resp. lines) no two of which are collinear (resp. concurrent). We remark that θ is the maximal number of points (lines) of $\mathbf{S}$, no two of which are collinear (resp. concurrent).

2.2. EXAMPLES OF OVALOIDS AND SPREADS. (a) If $\mathbf{S} = \mathbf{T}(k)$, then $\theta = k$. We see immediately that $\{x_{11}, x_{22}, \cdots, x_{kk}\}$ is an ovaloid and that $\{L_1, L_2, \cdots, L_k\}$ is a spread (we remark that $\mathbf{T}(k)$ possesses $k!$ ovaloids and 2 spreads).

(b) For $\mathbf{S} = \mathbf{Q}(4, q)$, we have $\theta = q^2 + 1$. Let $PG(3, q)$ be a hyperplane of $PG(4, q) \supset Q$ for which $PG(3, q) \cap Q = Q'$ is an elliptic quadric of $PG(3, q)$. Then Q' evidently is an ovaloid of $\mathbf{Q}(4, q)$. As $\mathbf{Q}(4, q)$, $q = 2^h$, is always self-dual ([9], [1]) there follows immediately that there exists also a spread of $\mathbf{Q}(4, q)$. Finally, we remark that $\mathbf{Q}(4, q)$, q odd, does not contain a spread [10].

If $\mathbf{S} = \mathbf{Q}(5, q)$, then $\theta = q^3 + 1$. We do not know if $\mathbf{Q}(5, q)$ possesses spreads or ovaloids.

(c) For $\mathbf{S} = \mathbf{H}(3, q)$, $q = p^{2h}$, we have $\theta = q\sqrt{q} + 1$. Let $PG(2, q)$ be a plane of $PG(3, q) \supset H$ for which $PG(2, q) \cap H = H'$ is a non-singular Hermitian curve of $PG(2, q)$. Then H' evidently is an ovaloid of $\mathbf{H}(3, q)$. We do not know if there exists a spread of $\mathbf{H}(3, q)$.

If $\mathbf{S} = \mathbf{H}(4, q)$, $q = p^{2h}$, then $\theta = q^2\sqrt{q} + 1$. We do not know if $\mathbf{H}(4, q)$ possesses spreads or ovaloids.

(d) For $\mathbf{S} = \mathbf{O}(q)$, $q = 2^h$, we have $\theta = q^2$. Let $PG^{(1)}(2, q)$ be a plane of $PG(3, q) \supset PG(2, q) \supset O$ where $PG^{(1)}(2, q) \cap PG(2, q) = L$ has no point in common with O. Then $PG^{(1)}(2, q) - L$ evidently is an ovaloid of $\mathbf{O}(q)$. It is also evident that the q^2 lines of $PG(3, q)$, which are not contained in $PG(2, q)$ and meet O in a fixed point, constitute a spread of the 4-gonal configuration $\mathbf{O}(q)$.

3. 4-Gonal subconfigurations of a given 4-gonal configuration

3.1. Definitions. The 4-gonal configuration $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ is called a 4-gonal subconfiguration of the 4-gonal configuration $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$ if and only if $\mathbf{P}' \subset \mathbf{P}$, $\mathbf{B}' \subset \mathbf{B}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathbf{P}' \times \mathbf{B}')$. If $\mathbf{S}' \neq \mathbf{S}$, then we say that $\mathbf{S}'$ is a proper 4-gonal subconfiguration of $\mathbf{S}$. When the parameters of $\mathbf{S}$ are denoted by k, r, v, b, the parameters of $\mathbf{S}'$ are denoted by k', r', v', b'.

3.2. Theorem. *If $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ is a 4-gonal subconfiguration of the 4-gonal configuration $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, then $y \mathrm{I} L \mathrm{I} x$, with $x, y \in \mathbf{P}'$ and $L \in \mathbf{B}$, implies $L \in \mathbf{B}'$. Dually, $L \mathrm{I} x \mathrm{I} M$, with $L, M \in \mathbf{B}'$ and $x \in \mathbf{P}$, implies $x \in \mathbf{P}'$.*

Proof. Suppose that $y \mathrm{I} L \mathrm{I} x$, with $x, y \in \mathbf{P}'$ and $L \in \mathbf{B}$. Let $L' \neq L$ be a line of $\mathbf{S}'$ which is incident with x. From (iii) in the definition of 4-gonal configuration there follows immediately that there are a unique point $x' \in \mathbf{P}'$ and a unique line $L'' \in \mathbf{B}'$, such that $y \mathrm{I}' L'' \mathrm{I}' x' \mathrm{I}' L'$. As $y \mathrm{I} L'' \mathrm{I} x' \mathrm{I} L'$ and $y \mathrm{I} L \mathrm{I} x \mathrm{I} L'$, it follows that $x = x'$ and $L = L''$. So we conclude that $L \in \mathbf{B}'$.

Corollaries. C1. *If $L \in \mathbf{B}$ then there are three possibilities*:

(a) $\nexists x \in \mathbf{P}' \,\|\, x \mathrm{I} L$; (b) $\exists!\, x \in \mathbf{P}' \,\|\, x \mathrm{I} L$; (c) $L \in \mathbf{B}'$.

C2. *If $x \in \mathbf{P}$ then there are three possibilities*:

(a) $\nexists L \in \mathbf{B}' \,\|\, x \mathrm{I} L$; (b) $\exists!\, L \in \mathbf{B}' \,\|\, x \mathrm{I} L$; (c) $x \in \mathbf{P}'$.

C3. *If $\mathbf{S}'$ is a proper 4-gonal subconfiguration of $\mathbf{S}$, then $\mathbf{P} \neq \mathbf{P}'$ and $\mathbf{B} \neq \mathbf{B}'$ (i.e. $v' < v$ and $b' < b$).*

4. THE CASE $k = k', r' < r$

4.1. THEOREM. *Suppose that the 4-gonal configuration* $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, *with parameters* k, r, v, b, *has a 4-gonal subconfiguration* $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$, *with parameters* k, r', v', b' $(r' < r)$. *Then* $\mathbf{S}'$ *possesses an ovaloid and* $r' - 1 \leq (r - 1)/(k - 1)$.

Proof. We consider a point $x \in \mathbf{P} - \mathbf{P}'$. From C2. there follows: $\nexists L \in \mathbf{B}' \| x \mathrm{I} L$ (C2'.). Let $L_1, L_2, \cdots, L_r$ be the r lines of $\mathbf{S}$ which are incident with x. Taking account of C1. and C2'., it follows that L_i, $i = 1, 2, \cdots, r$, is incident with at most one point of $\mathbf{S}'$. Suppose that $L_{i_1}, L_{i_2}, \cdots, L_{i_\alpha}$, $\{i_1, i_2, \cdots, i_\alpha\} \subset \subset \{1, 2, \cdots, r\}$, are the lines which are incident with one point of $\mathbf{S}'$. If $x_{i_j} \mathrm{I} L_{i_j}$, $x_{i_j} \in \mathbf{P}'$ and $j = 1, 2, \cdots, \alpha$, then from the definition of 4-gonal configuration there follows immediately that $\{x_{i_1}, x_{i_2}, \cdots, x_{i_\alpha}\} = O$ is a set of α points of $\mathbf{S}'$ no two of which are collinear. Hence the configuration $\mathbf{S}'$ contains $\alpha r'$ distinct lines which are incident with one of the points of O.

Next we consider an arbitrary line L' of $\mathbf{S}'$. There are a unique line $L'' \in \mathbf{B}$ and a unique point $x' \in \mathbf{P}$ such that $x \mathrm{I} L'' \mathrm{I} x' \mathrm{I} L'$. As $x' \in \mathbf{P}'$, it follows that $x' \in O$. So the line L' is incident with one of the points of O.

From the preceding there follows immediately that $\alpha r' = b' = r'(kr' - - k - r' + 2)$ or $\alpha = kr' - k - r' + 2$. Hence O is an ovaloid of $\mathbf{S}'$. Moreover we have $\alpha = kr' - k - r' + 2 \leq r$ or $r' - 1 \leq (r - 1)/(k - 1)$.

COROLLARIES. C4. $r \geq k$; $r = k$ implies $r' = 2$.

C5. If $k > 2$ then we have $r' \leq k$; $r' = k > 2$ implies $r - 1 = (k - 1)^2$.

Proof. From C4. there follows that $r > 2$, and so we have $r - 1 \leq \leq (k - 1)^2$ (1.2.). Consequently $r' - 1 \leq (r - 1)/(k - 1) \leq k - 1$ or $r' \leq k$.

If $r' = k > 2$ then $k - 1 = r' - 1 \leq (r - 1)/(k - 1)$, and so $r - 1 \geq \geq (k - 1)^2$. From the preceding there follows immediately that $r - 1 = = (k - 1)^2$.

REMARK. When $k = 2$ it is easy to prove that $2 \leq r' < r$ is the only restriction for r'.

C6. Suppose that $r' > 2$ and $k > 2$. Then $\sqrt{k - 1} \leq r' - 1 \leq k - 1$ (F) and $(k - 1)^{3/2} \leq r - 1 \leq (k - 1)^2$ (F').

Proof. From $r' > 2$ and $k > 2$ there follows that $r' \leq k$ and $(r' - 1)^2 \geq \geq k - 1$. So we have $\sqrt{k - 1} \leq r' - 1 \leq k - 1$.

Next we remark that $k - 1 \leq (r' - 1)^2 \leq (r - 1)^2/(k - 1)^2$ or $(k - 1)^3 \leq \leq (r - 1)^2$. As $k > 2$ and $r > 2$ we also have $r - 1 \leq (k - 1)^2$. We conclude that $(k - 1)^{3/2} \leq r - 1 \leq (k - 1)^2$.

REMARK. Let $r - 1 = (k-1)^{3/2}$, $k > 2$ and $r' > 2$. Then $r' - 1 \leq$ $\leq (k-1)^{3/2}/(k-1)$ or $r' - 1 \leq \sqrt{k-1}$. From (F) there follows that $r' - 1 = \sqrt{k-1}$.

C7. If $\mathbf{S}'$ possesses a proper 4–gonal subconfiguration $\mathbf{S}''$ with parameters $k, r'', v'', b'', k > 2$, then $r'' = 2, r' = k$ and $r - 1 = (k-1)^2$.

Proof. There holds $r' \leq k$ (C5.) and $r'' - 1 \leq (r'-1)/(k-1)$. Hence $r'' - 1 \leq \frac{r'-1}{k-1} \leq \frac{k-1}{k-1}$ or $r'' \leq 2$. There results that $r'' = 2$ and $r' = k$. From C5. there follows immediately that $r - 1 = (k-1)^2$.

4.2. THEOREM. *Suppose that the 4–gonal configuration* $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, *with parameters* k, r, v, b, *has a proper 4–gonal subconfiguration* $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$, *with parameters* k, r', v', b', *and suppose moreover that* $r' - 1 < (r-1)/(k-1)$. Then $\mathbf{P}'$ *is the union of k disjoint ovaloids of* $\mathbf{S}'$.

Proof. From the Proof of 4.1. there follows that $r' - 1 < (r-1)/(k-1)$ if and only if $\mathbf{B}$ contains a line L which is incident with no point of $\mathbf{P}'$. If $x_j \mathrm{I} L$, $x_j \in \mathbf{P}$ and $j = 1, 2, \cdots, k$, then the points of $\mathbf{P}'$ which are collinear with x_j constitute an ovaloid of $\mathbf{S}'$ (see Proof of 4.1.). In this way we obtain k ovaloids of $\mathbf{S}'$. As $\mathbf{S}$ does not contain a triangle, these ovaloids evidently are disjoint. From $v' = k\theta'$, with $\theta' = kr' - k - r' + 2$ the number of points of an ovaloid of $\mathbf{S}'$, there follows that $\mathbf{P}'$ is the union of k disjoint ovaloids of $\mathbf{S}'$.

4.3. REMARK. Suppose that O is an ovaloid of the 4–gonal configuration $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, with parameters k, r, v, b. If $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ is a 4–gonal subconfiguration of $\mathbf{S}$, with $k' = k$, then it is easy to prove that $O \cap \mathbf{P}'$ is an ovaloid of $\mathbf{S}'$.

4.4. THEOREM. *Let* $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ *be a substructure of the 4–gonal configuration* $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, *with parameters* $k, r, v, b\,(k > 2)$, *satisfying the following*:

(i) *every two distinct points of* $\mathbf{P}'$ *which are collinear in* $\mathbf{S}$ *are also collinear in* $\mathbf{S}'$;

(ii) *each element of* $\mathbf{B}'$ *is incident with k points of* $\mathbf{P}'$. *Then there are three possibilities*:

(*a*) *the elements of* $\mathbf{B}'$ *are lines which are incident with a same point of* $\mathbf{P}$, *and* $\mathbf{P}'$ *consists of the points of* $\mathbf{P}$ *which are incident with these lines*;

(*b*) $\mathbf{B}' = \emptyset$ *and* $\mathbf{P}'$ *is a set of points of* $\mathbf{P}$, *no two of which are collinear in* $\mathbf{S}$;

(*c*) $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ *is a 4–gonal subconfiguration of* $\mathbf{S}$ *with parameters* k, r', v', b'.

Proof. Evidently (i) and (ii) are fulfilled by (a), (b), (c). Now we show that there are no other possibilities.

Suppose that $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ satisfies (i), (ii) and is not of type (a), (b). Then there holds $\mathbf{B}' \neq \varnothing$ and $\mathbf{P}' \neq \varnothing$. Suppose that $L' \in \mathbf{B}'$. As $\mathbf{S}'$ is not of type (a) there exists a point $x' \in \mathbf{P}'$ such that $x' \not{\mathrm{I}}' L'$. Let x be the unique element of $\mathbf{P}$ and let L be the unique element of $\mathbf{B}$ for which $x' \mathrm{I} L \mathrm{I} x \mathrm{I} L'$. We remark that $x \in \mathbf{P}'$ (see (ii)). From $x, x' \in \mathbf{P}'$ and $x' \mathrm{I} L \mathrm{I} x$ there follows immediately that $L \in \mathbf{B}'$ (see (i)). And so axiom (iii) in the definition of 4-gonal configuration is satisfied by $\mathbf{S}'$. Now we show that also (i) and (ii) in definition 1.1. are satisfied.

First of all we remark that every line of $\mathbf{B}'$ is incident with k (> 2) points of $\mathbf{P}'$ and that two distinct lines of $\mathbf{B}'$ are incident with at most one point of $\mathbf{P}'$. Next we consider a point $x' \in \mathbf{P}'$ and we suppose that x' is incident with r' (> 0) lines of $\mathbf{B}'$. Let $y' \in \mathbf{P}'$ be a point which is not collinear with x' and call r'' the number of lines of $\mathbf{S}'$ which are incident with y' (as $\mathbf{S}'$ is not of type (a) or (b), such a point y' exists). If L' is a line of $\mathbf{B}'$ which is incident with x' (resp. y'), then there exists one and only one line of $\mathbf{B}'$ which is incident with y' (resp. x') and which is concurrent with L'. There follows immediately that $r' = r''$. In the same way we can prove that r' is the number of lines of $\mathbf{B}'$ which are incident with any point of $\mathbf{P}'$ which is collinear with x' but which is not collinear with y'. Finally we consider a point $z' \in \mathbf{P}'$ which is collinear with x' and y'. We have to consider two cases.

(a) Let us suppose that $r' = 1$. The line which is incident with x' (resp. y') and z' is denoted by L' (resp. L''). Suppose that L is a line of $\mathbf{B}'$ with $L \notin \{L', L''\}$. Then $x' \not{\mathrm{I}} L$, $y' \not{\mathrm{I}} L$ (since $r' = r'' = 1$). Since there exists a line of $\mathbf{B}'$ which is incident with x' (resp. y') and which is concurrent with L, there results that L and L' (resp. L and L'') are concurrent (taking account of $r' = r'' = 1$). So we conclude that $z' \mathrm{I} L$. Consequently $\mathbf{S}'$ is of type (a), a contradiction.

(b) Let us suppose that $r' > 1$. We consider a line L of $\mathbf{B}'$ which is incident with x' and which is not incident with z'. As $k > 2$ there exists a point $u' \in \mathbf{P}' - \{x'\}$ which is incident with L and which is not collinear with y'. Such a point u' is not collinear with z' or y'. There follows: number of lines of $\mathbf{S}'$ which are incident with z' = number of lines of $\mathbf{S}'$ which are incident with u' = number of lines of $\mathbf{S}'$ which are incident with $y' = r'$.

Consequently every point of $\mathbf{P}'$ is incident with r' (≥ 2) lines of $\mathbf{B}'$ and two distinct points of $\mathbf{P}'$ are incident with at most one line of $\mathbf{S}'$. So we conclude that $\mathbf{S}'$ is a 4-gonal subconfiguration of $\mathbf{S}$ with parameters k, r', v', b'.

REMARK. Consider the 4-gonal configuration $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \in)$, where $\mathbf{P} = \{1, 2, 3, 4, 5, 6\}$ and $\mathbf{B} = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$ (here we have $k = 2$, $r = 3$, $v = 6$, $b = 9$). Then the substructure $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \in)$, where $\mathbf{P}' = \{1, 2, 4, 5, 6\}$ and $\mathbf{B}' = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}\}$, satisfies (i), (ii) and is not of type (a), (b), (c). We conclude that condition $k > 2$, in the statement of Theorem 4.4., cannot be deleted.

4.5. EXAMPLES. (a) Let $\mathbf{P} = \{x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_r\}$ and $\mathbf{B} = \{L_{ij} \parallel i, j = 1, 2, \cdots, r\}$, where $r \geq 2$. Incidence is defined as follows: $L_{ij} \mathrm{I} x_l \Longleftrightarrow i = l$, $L_{ij} \mathrm{I} y_l \Longleftrightarrow j = l$. Then $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$ is a 4-gonal configuration with parameters $k = 2$, $r = r$, $v = 2r$, $b = r^2$. This configuration is denoted by $\mathbf{T}^*(r)$ and is the dual of the configuration $\mathbf{T}(r)$. Then the structure $\mathbf{T}^*(r') = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$, with $2 \leq r' \leq r$, $\mathbf{P}' = \{x_1, \cdots, x_{r'}, y_1, \cdots, y_{r'}\}$, $\mathbf{B}' = \{L_{ij} \parallel i, j = 1, \cdots, r'\}$, $\mathrm{I}' = \mathrm{I} \cap (\mathbf{P}' \times \mathbf{B}')$, evidently is a 4-gonal subconfiguration of $\mathbf{S}$ with parameters $k' = 2$, $r' = r'$, $v' = 2r'$, $b' = r'^2$.

(b) For $\mathbf{S} = \mathbf{Q}(4, q)$ we have $k = r = q + 1$. If $\mathbf{S}'$ is a proper 4-gonal subconfiguration of $\mathbf{Q}(4, q)$ with parameters $k' = q + 1$, r', v', b', then necessarily $r' = 2$ (C4.). It is easy to prove that $\mathbf{Q}(4, q)$ possesses such 4-gonal subconfigurations. Next, let $\mathbf{Q}^*(4, q) = \mathbf{W}(q)$ be the dual of $\mathbf{S}$. Then it is possible to prove that $\mathbf{W}(q)$ possesses proper 4-gonal subconfigurations, with $k' = q + 1$ (then necessarily $r' = 2$), if and only if q is even, i.e. if and only if $\mathbf{W}(q)$ is isomorphic to $\mathbf{Q}(4, q)$ (cfr. [1]).

For $\mathbf{S} = \mathbf{Q}(5, q)$ we have $k = q + 1$, $r = q^2 + 1$, and so $r - 1 = (k - 1)^2$. If $\mathbf{S}'$ is a proper 4-gonal subconfiguration of $\mathbf{Q}(5, q)$ with $k' = q + 1$, then necessarily $r' \leq q + 1$. If $r' > 2$ then we have $r' \geq \sqrt{q} + 1$ (cfr. C6.). Let $PG(4, q)$ be a hyperplane of $PG(5, q) \supset Q$ for which $PG(4, q) \cap Q = Q'$ is a non-singular hyperquadric of index 2 of $PG(4, q)$. Then $\mathbf{Q}'(4, q)$ is a proper 4-gonal subconfiguration of $\mathbf{Q}(5, q)$ with $k' = q + 1$ and $r' = q + 1$. Consequently $\mathbf{Q}'(4, q)$ possesses ovaloids. We remark that in this case $r' = k$ (cfr. C5.). From the preceding there also follows that $\mathbf{Q}'(4, q)$, and consequently $\mathbf{Q}(5, q)$, possesses 4-gonal subconfigurations with parameters $q + 1$, 2, $(q + 1)^2$, $2(q + 1)$ (cfr. C7.).

(c) As $r < k$ the configuration $\mathbf{H}(3, q)$, $q = p^{2h}$, has no proper 4-gonal subconfigurations with $k' = q + 1$ (C4.).

For $\mathbf{S} = \mathbf{H}(4, q)$, $q = p^{2h}$, we have $k = q + 1$, $r = 1 + q\sqrt{q}$, and so $r - 1 = (k - 1)^{3/2}$ (cfr. C6.). If $\mathbf{S}'$ is a proper 4-gonal subconfiguration of $\mathbf{H}(4, q)$ with $k' = q + 1$, then necessarily $r' = 2$ or $r' = \sqrt{q} + 1$ (see remark of C6.). Let $PG(3, q)$ be a hyperplane of $PG(4, q) \supset H$ for which $PG(3, q) \cap H = H'$ is a non-singular Hermitian primal of $PG(3, q)$. Then $\mathbf{H}'(3, q)$ is a proper 4-gonal subconfiguration of $\mathbf{H}(4, q)$ with $k' = q + 1$ and $r' = \sqrt{q} + 1$. Consequently $\mathbf{H}'(3, q)$ possesses ovaloids. It is not difficult to prove that there does not exist 4-gonal subconfigurations of $\mathbf{H}(4, q)$ with $k' = q + 1$ and $r' = 2$.

(d) We shall prove that $\mathbf{O}(q)$, $q = 2^h$ and $h > 1$, does not possess a proper 4-gonal subconfiguration with $k' = q$ and $r' > 2$. Suppose the contrary. Then from C6. there follows that $(k - 1)^{3/2} \leq r - 1 \leq (k - 1)^2$ or $(q - 1)^{3/2} \leq q + 1 \leq (q - 1)^2$, a contradiction. Finally we shall construct a 4-gonal subconfiguration $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$ of $\mathbf{O}(q)$ with $k' = q$ and $r' = 2$. Let $PG^{(1)}(2, q)$ be a plane of $PG(3, q) \supset PG(2, q) \supset O$, where $PG^{(1)}(2, q) \cap PG(2, q) = L$ has two distinct points x, y in common with O. Define: $\mathbf{P}' = PG^{(1)}(2, q) - L$, $\mathbf{B}' = \{$lines of $PG^{(1)}(2, q)$ which are different from L and contain x or $y\}$, incidence is that of $PG(3, q)$. Then the configuration $\mathbf{S}'$ so defined evidently is a 4-gonal subconfiguration of $\mathbf{S}$, with parameters $k' = q, r' = 2, v' = q^2, b' = 2q$.

5. THE CASE $k' < k$, $r' < r$

5.1. THEOREM. *Suppose that the 4-gonal configuration* $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, *with parameters* k, r, v, b, *has a 4-gonal subconfiguration* $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$, *with parameters* k', r', v', b' $(k' < k, r' < r)$. *Then* $(k' - 1)(r' - 1)^2 < (r - 1)(k - 1)$ *and dually* $(r' - 1)(k' - 1)^2 < (r - 1)(k - 1)$.

Proof. Suppose that $L' \in \mathbf{B}', x \mathrm{I} L', x \notin \mathbf{P}'$. Let $L \in \mathbf{B} - \{L'\}$ and $x \mathrm{I} L$ (from C2. there follows that $L \notin \mathbf{B}'$). Then we prove that $\mathbf{P}'$ does not contain an element which is incident with L.

Suppose a moment that $x' \in \mathbf{P}'$ and $x' \mathrm{I} L$. As $\mathbf{S}'$ is a 4-gonal configuration, there exists an element $y' \in \mathbf{P}'$ which is incident with L' and collinear with x'. There follows that $\mathbf{S}$ possesses a triangle (with vertices x, x', y'), a contradiction. So we conclude that $\mathbf{P}'$ does not contain an element which is incident with L.

Next, let $y \mathrm{I} L$ and $x \neq y$. From the preceding and from C2. there follows immediately that y is incident with at most one line of $\mathbf{S}'$. So the number of elements of $\mathbf{B}' - \{L'\}$, which are concurrent with a line of $\mathbf{B}$ which is incident with x, is not greater than $k'(r' - 1) + (k - 1)(r - 1)$. Further we remark that each element of $\mathbf{B}' - \{L'\}$ is concurrent with one and only one line of $\mathbf{B}$ which is incident with x. There results that $|\mathbf{B}' - \{L'\}| \leq$ $\leq k'(r' - 1) + (k - 1)(r - 1)$ or $r'(k'r' - k' - r' + 2) - 1 \leq k'(r' - 1) +$ $+ (k - 1)(r - 1)$. Hence $(k' - 1)(r' - 1)^2 \leq (r - 1)(k - 1)$.

Suppose a moment that $(k' - 1)(r' - 1)^2 = (r - 1)(k - 1)$. If $L \in \mathbf{B} -$ $- \{L'\}$, $x \mathrm{I} L$, $y \mathrm{I} L$, $x \neq y$, then y is incident with exactly one line M' of $\mathbf{S}'$. From the first part of the proof there follows that $\mathbf{P}'$ does not contain an element which is incident with a line $M \in \mathbf{B} - \{M'\}$, where $y \mathrm{I} M$. Hence y is collinear with exactly k' points of $\mathbf{P}'$. As each element of $\mathbf{P}'$ is collinear with one and only one point which is incident with L, there results $|\mathbf{P}'| = |\{\text{all } x \in \mathbf{P} \| x \mathrm{I} L\}| k' = kk'$ or $k'(k'r' - k' - r' + 2) = kk'$ or $k - 1 = (k' - 1)(r' - 1)$. Consequently $r' - 1 = r - 1$ or $r = r'$, a contradiction. We conclude that $(k' - 1)(r' - 1)^2 < (r - 1)(k - 1)$.

COROLLARIES. C8. $(k' - 1)^3 (r' - 1)^3 < (r - 1)^2 (k - 1)^2$.

C9. Suppose that $r' > 2$ and $k' > 2$. Then $(r' - 1)^5 < (k - 1)^6$ and dually $(k' - 1)^5 < (r - 1)^6$.

Proof. From $r' > 2$ and $k' > 2$ there follows that $r' - 1 \leq (k' - 1)^2$ or $\sqrt{r' - 1} \leq k' - 1$. So $\sqrt{r' - 1}\,(r' - 1)^2 < (r - 1)(k - 1)$. As $r > 2$ and $k > 2$ we have also $r - 1 \leq (k - 1)^2$. Hence $\sqrt{r' - 1}\,(r' - 1)^2 <$ $< (k - 1)^3$ or $(r' - 1)^5 < (k - 1)^6$.

C10. Suppose that $k = r = s$ and $k' = r' = s'$ $(s' < s)$. Then $(s - 1)^2 > (s' - 1)^3$.

5.2. THE PARTICULAR CASE $k = r = s$, $k' = r' = s'$ $(s' < s)$. *If the 4-gonal configuration* $\mathbf{S} = (\mathbf{P}, \mathbf{B}, \mathrm{I})$, *with* $k = r = s$, *has a proper 4-gonal subconfiguration* $\mathbf{S}' = (\mathbf{P}', \mathbf{B}', \mathrm{I}')$, *with* $k' = r' = s'$ *and* $s' \geq 13$, *then there holds* $(s - 1)^2 > 3(s' - 1)^3$.

Proof. If x (resp. L) is a point (resp. line) of $\mathbf{P} - \mathbf{P}'$ (resp. $\mathbf{B} - \mathbf{B}'$) which is not incident with a line (resp. point) of $\mathbf{S}'$, then the number of points (resp. lines) of $\mathbf{S}'$ which are collinear (resp. concurrent) with x (resp. L) is denoted by α_x (resp. β_L). We call α (resp. β) the maximum value of α_x (resp. β_L).

Let x be a point of $\mathbf{P} - \mathbf{P}'$ which is not incident with a line of $\mathbf{S}'$ and let $\alpha_x = \alpha$. Now it is not difficult to prove that $|\mathbf{B}'| = \alpha s' + \sum_L \beta_L$, where the summation runs over the lines L of $\mathbf{B}$ which are incident with x and which are not incident with a point of $\mathbf{P}'$ (the number of lines L equals $s - \alpha$). Consequently $|\mathbf{B}'| \leq \alpha s' + (s - \alpha)\beta$ or $s'((s' - 1)^2 + 1) \leq \alpha s' + (s - \alpha)\beta$ (1). Dually $\beta s' + (s - \beta)\alpha \geq s'((s' - 1)^2 + 1)$ (2). Summation of (1) and (2) gives: $\alpha(s + s' - 2\beta) + \beta(s + s') \geq 2s'((s' - 1)^2 + 1)$ (3). We distinguish four cases.

(a) Suppose that $\beta \geq (s + s')/2$ and $\alpha \geq (s + s')/2$. As $s + s' - 2\beta \leq 0$ and $\alpha \geq (s + s')/2$, there follows from (3) that $\frac{s + s'}{2}(s + s' - 2\beta) + \beta(s + s') \geq 2s'((s' - 1)^2 + 1)$ or $(s + s')^2 \geq 4s'((s' - 1)^2 + 1)$. Consequently

$$((s - 1) + (s' - 1) + 2)^2 \geq 4(s' - 1)^3 + 4(s' - 1)^2 + 4(s' - 1) + 4, \quad \text{or}$$

$$(s - 1)^2 \geq 4(s' - 1)^3 + 3(s' - 1)^2 - 2(s - 1)(s' - 1) - 4(s - 1).$$

(b) Suppose that $\beta \leq (s + s')/2$ and $\alpha \leq (s + s')/2$. As $s + s' - 2\beta \geq 0$ and $\alpha \leq (s + s')/2$, there follows from (3) that $\frac{s + s'}{2}(s + s' - 2\beta) + \beta(s + s') \geq 2s'((s' - 1)^2 + 1)$ or $(s + s')^2 \geq 4s'((s' - 1)^2 + 1)$. Consequently

$$(s - 1)^2 \geq 4(s' - 1)^3 + 3(s' - 1)^2 - 2(s - 1)(s' - 1) - 4(s - 1).$$

(c) Suppose that $\alpha \leq (s + s')/2$ and $\beta \geq (s + s')/2$.

If $s - 1 \geq (s' - 1)^2$, then $(s - 1)^2 \geq (s' - 1)^4 > 4(s' - 1)^3 + 3(s' - 1)^2$ (taking account of $s' \geq 13 > 5$). Consequently

$$(s - 1)^2 \geq 4(s' - 1)^3 + 3(s' - 1)^2 - 2(s - 1)(s' - 1) - 4(s - 1).$$

Next, let $s - 1 \leq (s' - 1)^2$. We prove that in this case $\alpha \geq s'$. Suppose the contrary. If L is a line of $\mathbf{B} - \mathbf{B}'$ which is not incident with a point of $\mathbf{S}'$, then it is not difficult to prove that $|\mathbf{P}'| = \beta_L s' + \sum_x \alpha_x$, where the summation runs over the points x of $\mathbf{P}$ which are incident with L and which are not incident with a line of $\mathbf{B}'$. Consequently $s'((s' - 1)^2 + 1) \leq \beta_L s' +$

$+ (s - \beta_L)\alpha < \beta_L s' + (s - \beta_L) s'$ or $s'((s'-1)^2+1) < ss'$. There results $s - 1 > (s'-1)^2$, a contradiction. We conclude that $\alpha \geq s'$. From $\beta(s'-\alpha) + s\alpha \geq s'((s'-1)^2+1)$ (see (2)), $s' - \alpha \leq 0$ and $\beta \geq (s+s')/2$ there follows that $\frac{s+s'}{2}(s'-\alpha) + s\alpha \geq s'((s'-1)^2+1)$ or $\alpha(s-s') + s'^2 + ss' \geq 2s'((s'-1)^2+1)$. As $\alpha \leq (s+s')/2$ there holds $\frac{s+s'}{2}(s-s') + s'^2 + ss' \geq 2s'((s'-1)^2+1)$ or $(s+s')^2 \geq 4s'((s'-1)^2+1)$. So we obtain again

$$(s-1)^2 \geq 4(s'-1)^3 + 3(s'-1)^2 - 2(s-1)(s'-1) - 4(s-1).$$

(d) If $\alpha \geq (s+s')/2$ and $\beta \leq (s+s')/2$, then again

$$(s-1)^2 \geq 4(s'-1)^3 + 3(s'-1)^2 - 2(s-1)(s'-1) - 4(s-1)$$

(dual of (c)).

We conclude that in all the possible cases

$$(s-1)^2 \geq 4(s'-1)^3 + 3(s'-1)^2 - 2(s-1)(s'-1) - 4(s-1).$$

Now we have to distinguish two cases.

1) $s - 1 > (s'-1)^2/2$. As $s' \geq 13$ there holds $3(s'-1)^2 \leq (s'-1)^4/4$, and so $(s-1)^2 > (s'-1)^4/4 \geq 3(s'-1)^3$.

2) $s - 1 \leq (s'-1)^2/2$. Then $3(s'-1)^2 \geq 6(s-1)$ and $(s'-1)^3 \geq 2(s-1)(s'-1)$. So $(s-1)^2 \geq 3(s'-1)^3 + 2(s-1)(s'-1) + 6(s-1) - 2(s-1)(s'-1) - 4(s-1)$ or $(s-1)^2 \geq 3(s'-1)^3 + 2(s-1)$. Consequently $(s-1)^2 > 3(s'-1)^3$.

We conclude that $(s-1)^2 > 3(s'-1)^3$ when $s' \geq 13$.

5.3. EXAMPLES. (a) Let Q' be a non-singular hyperquadric of index 2 of the projective space $PG(4, q)$ over the Galois field $GF(q)$. Now we consider the extension $GF(q^n)$ $(n > 1)$ of the field $GF(q)$ and also the corresponding extension $PG(4, q^n)$ (resp. Q) of $PG(4, q)$ (resp. Q') (we remark that Q is a non-singular hyperquadric of index 2 of the projective space $PG(4, q^n)$). Then the 4-gonal configuration $\mathbf{Q}'(4, q)$ is a proper 4-gonal subconfiguration of $\mathbf{Q}(4, q^n)$. In this case we have $k = r = q^n + 1$, $k' = r' = q + 1$ (q is a prime power and $n > 1$).

(b) Consider an irreducible conic C' of the plane $PG(2, q) \subset PG(3, q)$, where $q = 2^h$. If x is the nucleus of C', then $C' \cup \{x\} = O'$ is an oval of $PG(2, q)$. Let $GF(q^n)$ $(n > 1)$ be an extension of the field $GF(q)$ and let $PG(3, q^n)$ (resp. $PG(2, q^n)$, resp. C) be the corresponding extension of $PG(3, q)$ (resp. $PG(2, q)$, resp. C'). The nucleus of the irreducible conic C evidently is the point x. The oval $C \cup \{x\}$ of $PG(2, q^n)$ is denoted by O. Then the 4-gonal configuration $\mathbf{O}'(q)$ is a proper 4-gonal subconfiguration of $\mathbf{O}(q^n)$. In this case we have $k = q^n$, $r = q^n + 2$, $k' = q$, $r' = q + 2$ ($q = 2^h$ and $n > 1$).

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